Washington West Supervisory Union

K-12 Mathematics Curriculum

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Introduction

Acknowledgements
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Many thanks to these outstanding individuals not only for assisting in the development of this document, but for providing professional development to their colleagues in support of instructional practices aligned with the new Common Core standards.

Introduction to Mathematics Curriculum

The Common Core State Standards (CCSS) were adopted in Vermont in 2010. Since that time, the WWSU CCSS Mathematics Leadership Team has engaged in the process of understanding the "shifts" that these standards will require for instruction, and in the development of a curriculum document to reflect those changes. This document represents the standards and includes "content elaboration" to provide a greater context for those using the document to plan for instruction.

CCSS is very focused, and consequently we see less breadth and more depth within each grade, allowing teachers to spend more time on the most critically important concepts. CCSS represents content up to High School Algebra II and calls for all students to reach that level of mathematics during their time in school. Harwood's Program of Studies can provide more information about additional course offerings beyond Algebra II. Also included in the document is a *WWSU Computational Fluency Map* to solidify the expectations as to when certain skills are to be mastered, and "learning progressions" which help to identify the developmental continuum students progress through as they come to deeply understand mathematical concepts. Finally, our mathematics program materials are referenced within the document to help make the connections from the standards to the instructional materials used across the SU (Grades K-8).

As with all curricula, this written document is a guide to our standards-based instruction and as such is a working document to be refined and/or adapted as warranted by periodic review and evaluation.

Sheila Soule, Director
Curriculum and Assessment
October 2013

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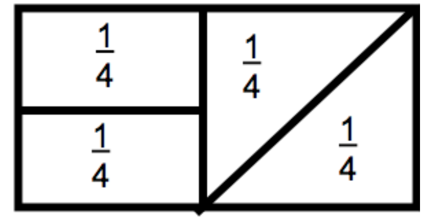
Mathematics Curriculum
K-8 Geometry

Domain	**Geometry Kindergarten**	**Geometry Grade 1**	**Geometry Grade 2**
Cluster	***Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).***		
Standards	**K.G.1.** Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as above, below, beside, in front of, behind, and next to. **K.G.2.** Correctly name shapes regardless of their orientations or overall size. **K.G.3.** Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").	**1.G.1.** Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes to possess defining attributes.	**2.G.1.** Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces. Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.
Content Elaborations	1. Please note it can be easy for teachers to mistakenly use 2d terms for 3d shapes! So be sure to use "sphere" when talking about a ball, etc.		
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Investigations	**Standard 1:** Unit 1: *Who is in School Today?* Unit 2: *Counting and Comparing* Unit 3: *What Comes Next?* Unit 4: *Measuring and Counting* Unit 5: *Make a Shape, Build a Block* **Standard 2:** Unit 1: *Who is in School Today?*	**Standard 1:** Unit 2: *Making Shapes and Designing Quilts* Unit 4: *What Would You Rather Be?* Unit 9: *Blocks and Boxes*	**Standard 1:** Unit 1: *How Many of Each* Unit 2: *Making Shapes and Designing Quilts* Unit 4: *What Would You Rather Be?* Unit 5: *Fish Lengths and Animal Lengths* Unit 6: *Number Games and Crayon Puzzles*

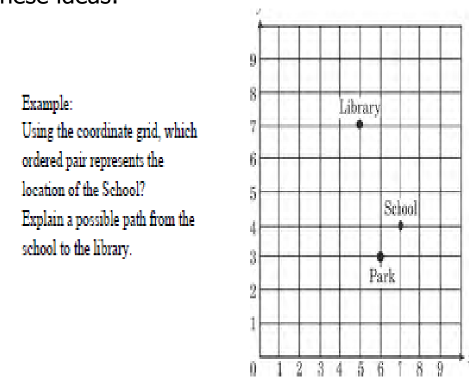
Domain	**Geometry Kindergarten**	**Geometry Grade 1**	**Geometry Grade 2**
Cluster	***Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).***		
	Unit 3: *What Comes Next?* Unit 5: *Make a Shape, Build a Block*		
Everyday Math		**Standard 1:** Unit 7: *Geometry and Attributes* Unit 10: *Year End Review*	**Standard 1:** *3.4, 4.3, 5.1, 5.3, 5.4, 5.5, 5.6, 5.7*

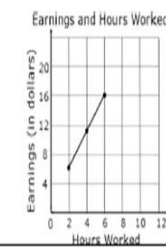
Domain	**Geometry Kindergarten**	**Geometry Grade 1**	**Geometry Grade 2**
Cluster	***Analyze, Compare, Compose and Decompose Shapes***	***Reason with shapes and their attributes***	
Standards	**K.G.4.** Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length). **K.G.5.** Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes. **K.G.6** Compose simple shapes to form larger shapes. For example, "Can you join these two triangles with full sides touching to make a rectangle?"	**1.G.2.** Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape. **1.G.3.** Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.	**2.G.2.** Partition a rectangle into rows of same-size squares and count to find the total number of them. **2.G.3.** Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words halves, thirds, half of, a third of, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.
Content Elaborations	**Standard 3:** Unit 5: *Make a Shape, Build a Block*	2. This is about putting shapes together and taking them apart--much like the work students do with number. Please note that while students may compose those three dimensional shapes, they are not responsible for knowing the names of those shapes (ie. cubes, right rectangular prisms, right circular cones, and right circular cylinders). 3. This is a student's first introduction to simple unit fractions. Please note that it is done only in the context of area models. We are using the words here, NOT the notation.	1. It's important to use correct terminology when talking about 2d versus 3d shapes (i.e., the flat surfaces in a 3d shape are called "faces" etc.) 2. Please note this links to students' work in the algebraic strand wherein they use arrays to explore the idea of adding equal groups, as well as laying the foundation for multiplicative reasoning and area of a shape. 3. Please note this is the first time students will have experience with thirds as well as with non-unit fractions.

Domain	**Geometry Kindergarten**	**Geometry Grade 1**	**Geometry Grade 2**
Cluster	***Analyze, Compare, Compose and Decompose Shapes***	***Reason with shapes and their attributes***	
			
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Investigations	**Standard 4:** Unit 5: *Make A Shape, Build A Block* **Standard 5:** Unit 5: *Make A Shape, Build a Block* **Standard 6:** Unit 5: *Make a Shape, Build a Block*	**Standard 2:** Unit 2: *Making Shapes and Designing Quilts* Unit 5: *Fish Lengths and Animal Lengths* Unit 9: *Blocks and Boxes* **Standard 3:** Unit 5: *Fish Lengths and Animal Lengths*	**Standard 2:** Unit 2: *Making Shapes and Designing Quilts* **Standard 3:** Unit 7: *Color, Shape, and Number Puzzles*
Everyday Math		**Standard 2:** Unit 3: *Visual Patterns, Number Patterns and Counting* Unit 7: *Geometry and Attributes* Unit 10: *Year End Review*	**Standard 2:** *4.7, 9.6, 9.7, 10.3* **Standard 3:** *8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7, 10.7*

Domain	**Geometry Kindergarten**	**Geometry Grade 1**	**Geometry Grade 2**
Cluster	***Analyze, Compare, Compose and Decompose Shapes***	***Reason with shapes and their attributes***	
		Standard 3: Unit 8*: Mental Arithmetic, Money and Fractions* Unit 9: *Place Values and Fractions*	

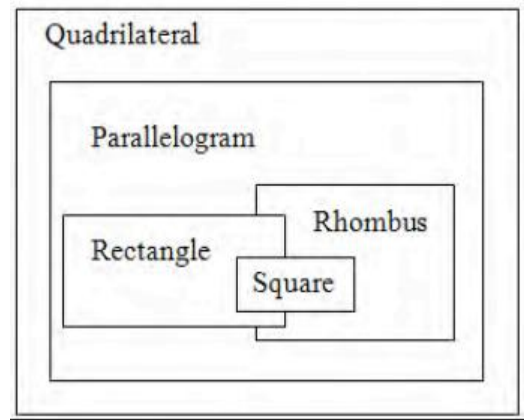
Domain	Geometry Grade 3	Geometry Grade 4	Geometry Grade 5
Cluster	***Reason with shapes and their attributes.***	***Draw and identify lines and angles, and classify shapes by properties of their lines and angles.***	***Graph points on the coordinate plane to solve real-world and mathematical problems.***
Standards	**3.G.1.** Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories. **3.G.2.** Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. For example, partition a shape into 4 parts with equal area, and describe the area of each part as 1/4 of the area of the shape.	**4.G.1.** Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures. **4.G.2.** Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles. **4.G.3.** Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.	**5.G.1.** Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., x-axis and x-coordinate, y-axis and y-coordinate). **5.G.2.** Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.

Domain	**Geometry Grade 3**	**Geometry Grade 4**	**Geometry Grade 5**
Cluster	***Reason with shapes and their attributes.***	***Draw and identify lines and angles, and classify shapes by properties of their lines and angles.***	***Graph points on the coordinate plane to solve real-world and mathematical problems.***
Content Elaborations	Show shapes in various positions on a paper.	Show all lines, shapes and angles in various positions on a paper.	The following examples (from http://www.ncpublicschools.org/docs/acre/standards/common-core-tools/unpacking/math/5th.pdf) might clarify these ideas:  Please note the following example links closely with the gr 5 operations and algebraic reasoning strand in the CCSS:

Domain	**Geometry** **Grade 3**	**Geometry** **Grade 4**	**Geometry** **Grade 5**
Cluster	***Reason with shapes and their attributes.***	***Draw and identify lines and angles, and classify shapes by properties of their lines and angles.***	***Graph points on the coordinate plane to solve real-world and mathematical problems.***
			Example: Sara has saved $20. She earns $8 for each hour she works. If Sara saves all of her money, how much will she have after working 3 hours? 5 hours? 10 hours? Create a graph that shows the relationship between the hours Sara worked and the amount of money she has saved. What other information do you know from analyzing the graph? Use the graph below to determine how much money Jack makes after working exactly 9 hours. 
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Investigations	**Standard 1:** Unit 4: *Perimeter, Angles, and Area* **Standard 2:** Unit 7: *Finding Fair Shares*	**Standard 1:** Unit 4: *Size, Shape, and Symmetry* **Standard 2:** Unit 4: *Size, Shape, and Symmetry* **Standard 3:**	**Standard 1:** Unit 8: *Growth Patterns* **Standard 2:** Unit 8: *Growth Patterns*

Domain	**Geometry** **Grade 3**	**Geometry** **Grade 4**	**Geometry** **Grade 5**
Cluster	***Reason with shapes and their attributes.***	***Draw and identify lines and angles, and classify shapes by properties of their lines and angles.***	***Graph points on the coordinate plane to solve real-world and mathematical problems.***
		Unit 4: *Size, Shape, and Symmetry*	
Everyday Math	**Standard 1:** Unit: *3.4, 6.5, 6.6, 6.9* **Standard 2:** Unit: *8.1, 8.3, 8.4, 8.5, 8.7* **Standard 3:** Unit: *3.4, 6.5, 6.6, 6.9, 8.1, 8.3, 8.4*	**Standard 1:** Unit 1: *Naming and Constructing Geometric Shapes* Unit 8: *Perimeter and Area* **Standard 2:** Unit 1: *Naming and Constructing Geometric Shapes* Unit 8: *Perimeter and Area* **Standard 3:** Unit 10: *Reflections and Symmetry*	**Standard 1:** Unit 9: *Coor, Area, Volume, Capacity* Unit 10: *Using Data and Algebra and Skills* Unit 12: *Prob, Ratios and Rates* **Standard 2:** Unit 9: *Coor, Area, Volume, Capacity* Unit 10: *Using Data and Algebra and Skills*

Domain	**Geometry Grade 3**	**Geometry Grade 4**	**Geometry Grade 5**
Cluster			***Classify two-dimensional figures into categories based on their properties.***
Standards			**5.G.3.** Understand that attributes belonging to a category of two- dimensional figures also belong to all subcategories of that category. For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles. **5.G.4.** Classify two-dimensional figures in a hierarchy based on properties.
Content Elaborations			4. "Hierarchy" is a term that refers to relationships within classifications of shapes. The following examples of student problems and work might be helpful (from: http://www.ncpublicschools.org/docs/acre/standards/common-core-tools/unpacking/math/5th.pdf): Given the following list of shapes, create a hierarchy that demonstrates the relationship among them: ***quadrilateral*** *– a four-sided polygon.* ***parallelogram*** *– a quadrilateral with two pairs of parallel and congruent sides.* ***rectangle*** *– a quadrilateral with two pairs of congruent, parallel sides and four right angles.* ***rhombus*** *– a parallelogram with all four sides equal in length.* ***square*** *– a parallelogram with four congruent sides and four right angles.*

Domain	**Geometry Grade 3**	**Geometry Grade 4**	**Geometry Grade 5**
Cluster			***Classify two-dimensional figures into categories based on their properties.***
			
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Investigations			**Standard 3:** Unit 5: *Measuring Polygons* **Standard 4:** Unit 5: *Measuring Polygons*
Everyday Math			**Standard 3:** *3.4, 6.5*

Domain	**Geometry** **Grade 3**	**Geometry** **Grade 4**	**Geometry** **Grade 5**
Cluster			***Classify two-dimensional figures into categories based on their properties.***
			Standard 4: Unit 3: *Geometry Explorations and the AT* Unit 4: *Division*

Domain	**Geometry Grade 6**	**Geometry Grade 7**	**Geometry Grade 8**
Cluster	***Solve real-world and mathematical problems involving area, surface area, and volume.***	***Draw, construct, and describe geometrical figures and describe the relationships between them.***	***Understand congruence and similarity using physical models, transparencies, or geometry software.***
Standards	**6.G.1.** Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems. **6.G.2.** Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas V = l w h and V = b h to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems. **6.G.3.** Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems. **6.G.4.** Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.	**7.G.1.** Solve problems involving scale drawings of geometric figures, such as computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale. **7.G.2.** Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle. **7.G.3.** Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.	**8.G.1.** Verify experimentally the properties of rotations, reflections, and translations: a. Lines are taken to lines, and line segments to line segments of the same length. b. Angles are taken to angles of the same measure. c. Parallel lines are taken to parallel lines. **8.G.2.** Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them. **8.G.3.** Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates. **8.G.4.** Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two- dimensional figures, describe a sequence that exhibits the similarity between them. **8.G.5.** Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of

Domain	**Geometry Grade 6**	**Geometry Grade 7**	**Geometry Grade 8**
Cluster	***Solve real-world and mathematical problems involving area, surface area, and volume.***	***Draw, construct, and describe geometrical figures and describe the relationships between them.***	***Understand congruence and similarity using physical models, transparencies, or geometry software.***
			triangles. For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.
Content Elaborations	It is very important for students to continue to physically manipulate materials and make connections to the symbolic and more abstract aspects of geometry. Exploring possible nets should be done by taking apart (unfolding) three-dimensional objects. This process is also foundational for the study of surface area of prisms. Building upon the understanding that a net is the two-dimensional representation of the object, students can apply the concept of area to find surface area. The surface area of a prism is the sum of the areas for each face. 1. In order to derive area of geometric figures students build upon their understanding how to find the area of familiar 2-dimensional shapes (such as trapezoid is made up of a rectangle and two triangle). 2. To find the volume, you can find look at it as finding the area of the base and then multiplying it by the height of the base or as three separate measurements (length, width & height). 2.&4. Students should be able to compute the	1. Find and use the scale factors of geometric figures (2-dimensional) to determine side lengths and area of similar figures. Computing actual lengths includes finding perimeter. 2. Given 3 angles, student can construct different triangles. Triangles will be similar, but not unique. Given 3 sides of a triangle, only 1 unique triangle can be constructed. Given 3 different lengths, a triangle can only be constructed if the sum of the measure of any two sides is greater than the length of the third side. (Triangle Inequality Theorem). 3. (Cross-sections)- Given a 3 dimensional figure and its cross-section, can identify the 2 dimensional figure.	This cluster interweaves the relationships of symmetry, transformations, and angle relationships to form understandings of similarity and congruence. Students should be able to appropriately label figures, angles, lines, line segments, congruent parts, and images (primes or double primes). 1. Line up congruent figures so that corresponding angles and sides match up. 2. Describe each transformation, step by step, that will prove congruence. 3-4. Dilation is a proportional enlargement or reduction of a figure resulting on similar figures.

Domain	**Geometry Grade 6**	**Geometry Grade 7**	**Geometry Grade 8**
Cluster	***Solve real-world and mathematical problems involving area, surface area, and volume.***	***Draw, construct, and describe geometrical figures and describe the relationships between them.***	***Understand congruence and similarity using physical models, transparencies, or geometry software.***
	volume of 3-dimensional figures with fractional measures. 3. Students will use the coordinate grid to find the length of a side vertically or horizontal (not diagonally - same x value or same y value).		
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Connected Math	**Standard 1:** Investigation 1-5: *Covering & Surrounding* **Standard 2:** CC Investigation 4: *Bits & Pieces III* **Standard 3:** ACE Investigation 2 #39: *Shapes & Designs* CC Investigation 3: *Bits & Pieces III* **Standard 4:** ACE Investigation 3 #39: *Covering & Surrounding* CC Investigation 4: *Bits & Pieces III*	**Standard 1:** Investigation 1-5: *Stretching and Shrinking* **Standard 1:** Investigation 4: *Comparing and Sealing* **Standard 2:** Investigation 1-4: *Filling and Wrapping* CC Investigation 4: *Filling and Wrapping* **Standard 3:** CC Investigations : *Filling and Wrapping*	**Standard 1:** Investigation 1-5: *Kaleidoscopes, Hubcaps, and Mirrors* CC Investigation 3: *Transformations* **Standard 2:** Investigation 3: *Kaleidoscopes, Hubcaps, and Mirrors* **Standard 3:** ACE Investigation 2 #24, #25, #32: *Kaleidoscopes, Hubcaps, and Mirrors* CC Investigation 3 and 5: *Transformations* **Standards 4, 5:** CC Investigation 4: *Geometry Topics in Kaleidoscopes, Hubcaps, and Mirrors*

Domain	**Geometry Grade 6**	**Geometry Grade 7**	**Geometry Grade 8**
Cluster		***Solve real-world and mathematical problems involving area, surface area, and volume.***	
Standards		**7.G.4.** Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle. **7.G.5.** Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure. **7.G.6.** Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.	
Content Elaborations		4: Radius, diameter, circumference and area of a circle, as well as the relationships between them are new concepts. 4-6: Students apply geometric terms, formulas and relationships to solve "real life" problems.	

Domain	**Geometry Grade 6**	**Geometry Grade 7**	**Geometry Grade 8**
Cluster		***Solve real-world and mathematical problems involving area, surface area, and volume.***	
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Connected Math		**Standard 4:** CC Investigation 4: *Filling and Wrapping* **Standard 5:** ACE Investigation 3 #22-24: *Stretching and Shrinking* CC Investigation 4: *Filling and Wrapping* **Standard 6:** Investigation 2, 3: *Stretching and Shrinking* Investigations 1-5: *Filling and Wrapping*	

Domain	Geometry Grade 6	Geometry Grade 7	Geometry Grade 8
Cluster			***Understand and apply the Pythagorean Theorem.***
Standards			**8.G.6.** Explain a proof of the Pythagorean Theorem and its converse. **8.G.7.** Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions. **8.G.8.** Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.
Content Elaborations			Previous understanding of triangles, such as the sum of two side measures is greater than the third side measure, angles sum, and area of squares, is furthered by the introduction of unique qualities of right triangles. Students should be given the opportunity to explore right triangles to determine the relationships between the measures of the legs and the measure of the hypotenuse. 8. The Pythagorean Theorem should be applied to finding the lengths of segments on a coordinate grid, especially those segments that do not follow the vertical or horizontal lines, as a means of discussing the determination of distances between points. Contextual situations, created by both the students and the teacher, that apply the Pythagorean theorem and its

Domain	**Geometry** **Grade 6**	**Geometry** **Grade 7**	**Geometry** **Grade 8**
Cluster			***Understand and apply the Pythagorean Theorem.***
			converse should be provided. For example, apply the concept of similarity to determine the height of a tree using the ratio between the student's height and the length of the student's shadow. From that, determine the distance from the tip of the tree to the end of its shadow and verify by comparing to the computed distance from the top of the student's head to the end of the student's shadow, using the ratio calculated previously. Challenge students to identify additional ways that the Pythagorean Theorem is or can be used in real world situations or mathematical problems, such as finding the height of something that is difficult to physically measure, or the diagonal of a prism.
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Connected Math			**Standard 6:** Investigation 3: *Looking for Pythagoras* **Standard 7:** Investigation 3 and 4: *Looking for Pythagoras*

Domain	**Geometry** **Grade 6**	**Geometry** **Grade 7**	**Geometry** **Grade 8**
Cluster			***Understand and apply the Pythagorean Theorem.***
			Standard 8: Investigation 2 and 3: *Looking for Pythagoras*

Domain	**Geometry Grade 6**	**Geometry Grade 7**	**Geometry Grade 8**
Cluster			***Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.***
Standards			**8.G.9.** Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.
Content Elaborations			No elaboration necessary.
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Connected Math			**Standard 9:** ACE Investigation 1 #47-49: *Kaleidoscopes, Hubcaps, and Mirrors* ACE Investigation 2 #28: *Kaleidoscopes, Hubcaps, and Mirrors* ACE Investigation 3 #24: *Kaleidoscopes, Hubcaps, and Mirrors* ACE Investigation 3 #18 –22, #25-26: *Looking for Pythagoras* ACE Investigation 4 #57-58: *Looking for Pythagoras* ACE Investigation 1 #55: *Say it with Symbols* ACE Investigation 3 #41: *Say it with Symbols*

Domain	**Geometry** **Grade 6**	**Geometry** **Grade 7**	**Geometry** **Grade 8**
Cluster			***Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.***
			ACE Investigation 4 #39: *Say it with Symbols* CC Investigations 4: *Geometry Topics in Kaleidoscopes, Hubcaps, and Mirrors*

Domain	**Measurement and Data** **Kindergarten**	**Measurement and Data** **Grade 1**	**Measurement and Data** **Grade 2**
Cluster	***Describe and compare measurable attributes.***	***Measure lengths indirectly and by iterating length units.***	***Measure and estimate lengths in standard units***
Standards	**K.MD.1.** Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object. **K.MD.2.** Directly compare two objects with a measurable attribute in common, to see which object has "more of" / "less of" the attribute, and describe the difference. *For example, directly compare the heights of two children and describe one child as taller/shorter.*	**1.MD.1.** Order three objects by length; compare the lengths of two objects indirectly by using a third object. **1. MD.2.** Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. *Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.*	**2.MD.1.** Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks and measuring tapes. **2.MD.2.** Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen. **2.MD.3.** Estimate lengths using units of inches, feet, centimeters, and meters. **2.MD.4.** Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.
Content Elaborations	1. For example, a child might understand that the length of the edges of a book as well as its thickness can be measured, but that the book also has weight. 2. None needed	1. For example, children may compare the length of objects to the length of a particular pencil, and order them from largest to smallest or smallest to largest. 2. For example, children might use the edges of inch-tiles to measure the length of objects. It's important for children to understand they are measuring the one-dimensional line that represents length with the one-dimensional line formed by the edge of whatever material they are using to measure.	1. None needed 2. This is the notion that the longer the measuring unit, the fewer the number of those units are used; conversely, the smaller the unit, the larger the total number used to measure the length of the same object. 3. Children do not need to estimate all objects with all units of measure;

Domain	Measurement and Data Kindergarten	Measurement and Data Grade 1	Measurement and Data Grade 2
Cluster	***Describe and compare measurable attributes.***	***Measure lengths indirectly and by iterating length units.***	***Measure and estimate lengths in standard units***
			rather, appropriate units of lengths should be used. 4. This is also related to the idea of comparing and finding a difference.
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Investigations	**Standard 1:** Unit 2: *Counting and Comparing* Unit 4: *Measuring and Counting* Unit 6: *How Many Do You Have?* **Standard 2:** Unit 2: *Counting and Comparing* Unit 4: *Measuring and Counting*	**Standard 1:** Unit 5: *Fish Lengths and Animal Lengths* **Standard 2:** Unit 5: *Fish Lengths and Animal Lengths*	**Standard 1:** Unit 9: *Measuring Length and Time* **Standard 2**: Unit 9: *Measuring Length and Time* **Standard 3**: Unit 9: *Measuring Length and Time* **Standard 4**: Unit 9: *Measuring Length and Time*
Everyday Math		**Standard 1:** *2.7, 4.2, 4.4, 4.5 6.6* **Standard 2:** *4.2, 4.3, 4.4, 4.5, 4.6, 4.7, 6.6, 6.11, 9.5*	**Standard 1:** *4.7, 7.6, 9.1, 9.2, 9.3, 9.4* **Standard 2:** *4.7, 7.6, 9.2, 9.3* **Standard 3:** *9.1, 9.2, 9.3*

Domain	**Measurement and Data** **Kindergarten**	**Measurement and Data** **Grade 1**	**Measurement and Data** **Grade 2**
Cluster	***Describe and compare measurable attributes.***	***Measure lengths indirectly and by iterating length units.***	***Measure and estimate lengths in standard units***
			Standard 4: *7.6, 7.7, 7.8, 9.1, 9.2, 9.3*

Domain	**Measurement and Data** **Kindergarten**	**Measurement and Data** **Grade 1**	**Measurement and Data** **Grade 2**
Cluster			***Relate addition and subtraction to length***
Standards			**2.MD.5** Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem. **2.MD.6** Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, ..., and represent whole-number sums and differences within 100 on a number line diagram.
Content Elaborations			
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Investigations			**Standard 5:** Unit 9: *Measuring Length and Time* **Standard 6:** Unit 1: *Counting, Coins and Combinations* Unit 3: *Stickers, Number Strings, Story Problems* Unit 6: *How Many Tens? How Many Ones?* Unit 8: *Partners, Teens and Paperclips*
Everyday Math			

Domain	**Measurement and Data** **Kindergarten**	**Measurement and Data** **Grade 1**	**Measurement and Data** **Grade 2**
Cluster	***Classify objects and count the number of objects in each category.***	***Tell and write time***	***Work with Time and Money***
Standards	**K.MD.3** Classify objects into given categories; count the numbers of objects in each category and sort the categories by count.	**1.MD.3.** Tell and write the time in hours and half hours using analog and digital clocks	**2.MD. 7.** Tell and write time from analog and digital clocks to the nearest five minutes, using a.m. and p.m. **2. MD. 8.** Solve word problems involving dollar bills, quarters, dimes, nickels, and pennies, using $ and ¢ symbols appropriately. Example: If you have 2 dimes and 3 pennies, how many cents do you have?
Content Elaborations	None needed	None needed	
Expectations for Learning (Benchmark Indicators)	All units		
Program Correlations			
Investigations	**Standard 3:** Unit 1: *Who's In School Today?* Unit 2: *Counting and Comparing* Unit 3: *What Comes Next?* Unit 4: *Measuring and Counting* Unit 5: *Make a Shape, Build a Block* Unit 6: *How Many Do You Have?* Unit 7: *Sorting and Surveys*	**Standard 3:** Unit 4: *What Would You Rather Be?* Unit 5: *Fish Lengths and Animal Lengths* Unit 6: *Number Games and Crayon Puzzles* Unit 7: *Color, Shape, and Number Puzzles* Unit 8: *Twos, Fives, Tens* Unit 9: *Blocks and Boxes*	**Standard 7**: All Units Unit 1: *Counting Coins and Combinations* Unit 2: *Shapes, Blocks and Symmetry* Unit 3: *Stickers, Number Strings, and Story Problems* Unit 4: *Pockets, Teeth and Favorite Things* Unit 5: *How Many Floors? How Many Rooms?* Unit 6: *How Many Tens? How Many Ones?* Unit 7: *Parts of a Whole, Parts of a Group* Unit 8: *Partners, Teams and Paper Clips* Unit 9: *Measuring Length and Time*

Domain	**Measurement and Data** **Kindergarten**	**Measurement and Data** **Grade 1**	**Measurement and Data** **Grade 2**
Cluster			***Relate addition and subtraction to length***
			Standard 8: Unit 1: *Counting Coins and Combinations* Unit 3: *Stickers, Number Strings, and Story Problems* Unit 4: P*ockets, Teeth and Favorite Things* Unit 5: *How Many Floors? How Many Rooms?* Unit 6: *How Many Tens? How Many Ones?* Unit 9: *Measuring Length and Time*
Everyday Math		**Standard 3:** *2.5, 2.6, 3.7, 3.8, 4.4, 4.8, 4.9, 4.10, 6.10, 6.11, 7.2, 8.1, 10.1, 10.2, 10.5*	**Standard 7:** Unit 1: *Numbers and Routines* Unit 3: *Place Value, Money and Time* Unit 12: *Review* **Standard 8:** Unit 1: *Numbers and Routines* Unit 3: *Place Value, Money and Time* Unit 4: *Addition and Subtraction* Unit 6: *Whole Number Operations and Stories* Unit 10: *Decimals and Place Value* Unit 11: *Whole Number Operations Revisited*

Domain	Measurement and Data Kindergarten	Measurement and Data Grade 1	Measurement and Data Grade 2
Cluster		***Represent and Interpret Data***	
Standards		**1.MD.4.** Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another. Students are only working with categorical data at this level. For example, they might collect favorite animals and sort them into meaningful groups, and interpret and work with the data as discussed in the standard.	**2.MD.9.** Generate measurement data by measuring lengths of several objects to the nearest whole unit, or by making repeated measurements of the same object. Show the measurements by making a line plot, where the horizontal scale is marked off in whole-number units. **2.MD.10.** Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put together, take-apart, and compare problems using information presented in a bar graph.
Content Elaborations			9. *Note: This is the first time students will be working with numerical data; this data is in reference to measuring length. All previous experiences with data in the Common Core have been categorical data.* For example, a group of children might all measure the same magic marker, and their results plotted on a line plot. A discussion might then arise about what accounts for any discrepancies, for example, which could lead to a conversation about precision and accuracy in measuring. 10. For example, a student should be able to create, interpret and analyze picture graphs and bar graph about a set of data. For example, students might add the number of pieces of data in each of the categories to find the total number of data points on the graph. They could also compare the number of data in two categories and calculate the difference between them. They might also take the total number of data points and desegregate it into the totals of

Domain	Measurement and Data Kindergarten	Measurement and Data Grade 1	Measurement and Data Grade 2
Cluster		***Represent and Interpret Data***	
			each category group.
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Investigations		**Standard 4:** Unit 1: *How Many of Each* Unit 3: *Solving Story Problems* Unit 4: *What Would You Rather Be?* Unit 5: *Fish Lengths and Animal Lengths* Unit 6: *Number Games and Crayon Puzzles* Unit 7: *Color, Shape, and Number Puzzles* Unit 8: *Twos, Fives, Tens* Unit 9: *Blocks and Boxes*	**Standard 9**: Unit 9: *Measuring Length and Time* **Standard 10**: Unit 4: *Pockets, Teeth and Favorite Things* Unit 5: *How Many Floors? How Many Rooms?*
Everyday Math		**Standard 4:** *1.7, 1.8, 1.12, 2.11, 2.13, 4.5, 4.7, 5.9, 6.12, 7.3, 7.4, 8.1, 9.2, 9.6, 10.1, 10.3*	**Standard 9:** Unit 7: *Patterns and Rules* Unit 10: *Decimals and Place Value* Unit 11: *Whole Number Operations Revisited* Unit 12: *Year End Review* **Standard 10:** Unit 3: *Place Value Money and Time* Unit 6: *Whole Number Operations and Stories* Unit 7: *Pattern and Rules* Unit 12: *Year End Review*

Domain	**Measurement and Data** **Grade 3**	**Measurement and Data** **Grade 4**	**Measurement and Data** **Grade 5**
Cluster	***Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.***	***Solve problems involving measurement and conversion of measurement from a larger unit to a smaller unit.***	***Convert like measurement units within a given measurement system.***
Standards	**3.MD.1.** Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, *e.g., by representing the problem on a number line diagram.* **3.MD.2.** Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l). Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.	**4.MD.1.** Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two column table. *For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36), ...* **4.MD.2.** Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale. **4.MD.3.** Apply the area and perimeter formulas for rectangles in real world and mathematical problems.	**5.MD.1.** Convert among different-sized standard measurement units within a given measurement system (e.g. convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.
Content Elaborations	1. None needed 2. Note: Liquid volume is not specifically addressed in Investigations.	1. Please note that this will involve some thinking related to the number system and powers of ten developed in the number and operations strand. Children will need to make sense of these exponential relationships through models and experiences.	1. Note: Not limited to just linear units of measure but also includes volume, area, units of time, etc. Again, please note that this work is related to and scaffolds work with ratio and functions that is developed in the operations and algebraic thinking strand.

Domain	**Measurement and Data** **Grade 3**	**Measurement and Data** **Grade 4**	**Measurement and Data** **Grade 5**
Cluster	***Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.***	***Solve problems involving measurement and conversion of measurement from a larger unit to a smaller unit.***	***Convert like measurement units within a given measurement system.***
		2. Note: Liquid volume is not specifically addressed in Investigations. 3. This involves understanding the impact an operation has on whole numbers versus fractions. Like Standard 1, keep in mind that this also involves conversions and thus scaffolds the idea of ratio. 4. Understanding of formulas is developed through models, experience, manipulatives, etc. Students will be able to calculate a given area, for example, but can also identify a situation that matches a given equation for an area.	
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Investigations	**Standard 1:** Unit 3: *Collections and Travel Stories: Addition, Subtraction, and the Number System (Ten-Minute Math)* Unit 5: *Equal Groups: Multiplication and Division (Ten-Minute Math)*	**Standard 1.**: Unit 4: *Size, Shape, and Symmetry* Unit 7: *Moving Between Solids and Silhouettes* Unit 9: *Penny Jars and Plant Growth* **Standard 2:** Unit 2: *Describing the Shape of the Data* Unit 4: *Size, Shape, and Symmetry*	**Standard 1:** Unit 6*: Decimals on Grids and Number Lines* Unit 8: *Growth Patterns*

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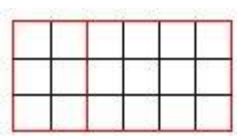
Domain	**Measurement and Data** **Grade 3**	**Measurement and Data** **Grade 4**	**Measurement and Data** **Grade 5**
Cluster	***Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.***	***Solve problems involving measurement and conversion of measurement from a larger unit to a smaller unit.***	***Convert like measurement units within a given measurement system.***
	Unit 7: *Finding Fair Shares: Fractions and Decimals (Ten-Minute Math)* **Standard 2:** Unit 9: *Solids and Boxes: 3-D Geometry and Measurement*	Unit 5: *Landmarks and Large Numbers* Unit 6: *Fraction Cards and Decimal Squares* Unit 7: *Moving Between Solids and Silhouettes* Unit 8: *How Many Packages? How Many Groups?*	
Everyday Math	**Standard 1:** *1.4, 1.13, 5.5, 5.12, 11.1* **Standard 2:** *9.10, 10.3, 10.4, 10.5, 10.8*	**Standard 1:** Unit 4: *Decimals and Their Use* Unit 5: *Big Numbers, Estimation, and Computation* Unit 8: *Perimeter and Area* Unit 11:*3-D Shapes, Weights, Volume, and Capacity* Unit 12: *Rates* **Standard 2:** Unit 2: *Using Numbers and Organizing Data* Unit 3: *Multiplication and Division; Number Sentences and Algebra* Unit 4: *Decimals and Their Use* Unit 5: *Big Numbers, Estimation, and Computation* Unit 6: *Division; Maps; and Measure of Angles* Unit 7: *Fractions and Their Use; Probability* Unit 8: *Perimeter and Area* Unit 9: *Fractions, Decimals, and Percents* Unit 11: *3-D Shapes, Weights, Volume and Capacity* Unit 12*: Rates* **Standard 3:** Unit 8: *Perimeter and Area* Unit 9: *Fractions, Decimals, and Percents* Unit 11*: 3-D Shapes, Weights, Volume and Capacity*	**Standard 1:** Unit 2: *Estimation and Computation* Unit 9: *Coordinates, Area, Volume, Capacity*

Domain	**Measurement and Data Grade 3**	**Measurement and Data Grade 4**	**Measurement and Data Grade 5**
Cluster	***Represent and interpret data.***		
Standards	**3. MD.3.** Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. *For example, draw a bar graph in which each square in the bar graph might represent pets.* **3. MD. 4.** Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters.	**4. MD.4.** Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Solve problems involving addition and subtraction of fractions by using information presented in line plots. *For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.*	**5. MD.2.** Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Use operations on fractions for this grade to solve problems involving information presented in line plots. *For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.*
Content Elaborations	3. A scaled graph is one that uses a single unit to represent an assigned value; for example, each box in a bar graph might represent 5 children. Please note that this standard speaks to kids doing multi-step problem. For example, they might be asked to find how many more kids like cheese pizza than like pepperoni pizza and mushroom pizza combined 4. none needed	4. This underscores the idea of subtraction as comparison. Use the tables for common addition and subtraction (Table 1 in the Common Core State Standards for Mathematics) as a guide to the types of problems students need to solve.	2. Students should use visual models to solve problems. Use the tables for common addition and subtraction, and multiplication and division situations (Table 1 and Table 2 in the Common Core State Standards for Mathematics) as a guide to the types of problems students need to solve.
Program Correlations			
Investigations	**Standard 3:** Unit 2: *Surveys and Line Plots: Data Analysis* **Standard 4:** Unit 2: *Surveys and Line Plots: Data Analysis*	**Standard 4:** Unit 6: *Fraction Cards and Decimal Squares* Unit 9: *Penny Jars and Plant Growth*	**Standard 2:** Unit 9: *How Long Can You Stand On One Foot*
Everyday Math	**Standard 3:** *1.5, 1.10, 5.2, 10.6, 10.7, 10.9, 11.1* **Standard 4:** *3.2, 3.3, 3.5, 5.7, 9.13, 10.7*	**Standard 4:** Unit 2: *Using Numbers and Organizing Data* Unit 7: *Fractions and Their Use; Probability* Unit 11: *3-D Shapes, Weights, Volume and Capacity*	**Standard 2:** Unit 2: *Estimation and Computation* Unit 6: *Using Data and Add and Sub Fractions* Unit 7: *Exponents and Negative Numbers*

Domain	**Measurement and Data Grade 3**	**Measurement and Data Grade 4**	**Measurement and Data Grade 5**
Cluster	***Geometric measurement: understand concepts of area and relate area to multiplication and to addition.***	***Geometric measurement: understand concepts of angle and measure angles.***	***Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.***
Standards	**3.MD.5.** Recognize area as an attribute of plane figures and understand concepts of area measurement. a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area. b. A plane figure which can be covered without gaps or overlaps by *n* unit squares is said to have an area of *n* square units. **3.MD.6.** Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units). **3.MD. 7.** Relate area to the operations of multiplication and addition. a. Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths. b. Multiply side lengths to find areas of rectangles with whole number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning. c. Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning. d. Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.	**4.MD.5.** Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement: a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles. b. An angle that turns through *n* one-degree angles is said to have an angle measure of *n* degrees. **4.MD.6.** Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure. **4.MD.7.** Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.	**5.MD.3.** Recognize volume as an attribute of solid figures and understand concepts of volume measurement. a. A cube with side length 1 unit, called a "unit cube," is said to have "one cubic unit" of volume, and can be used to measure volume. b. A solid figure which can be packed without gaps or overlaps using *n* unit cubes is said to have a volume of *n* cubic units. **5.MD.4.** Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units. **5.MD.5.** Relate volume to the operations of multiplication and addition and solve real world and mathematical problems involving volume. a. Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication. b. Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole number edge lengths in the context of solving real world and mathematical problems. c. Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.

Domain	**Measurement and Data** **Grade 3**	**Measurement and Data** **Grade 4**	**Measurement and Data** **Grade 5**
Cluster	***Geometric measurement: understand concepts of area and relate area to multiplication and to addition.***	***Geometric measurement: understand concepts of angle and measure angles.***	***Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.***
Content Elaborations	5a. none needed 5b. Note that children are now iterating 2-dimensional units of measure (i.e. the square face of the tile, rather than its edge, and that iteration still means the same thing--one unit ends where the next begins (no gaps or overlaps). 6. none needed 7a. Tiling means iterating a 2d unit shape, i.e. covering a rectangular area with square tiles. 7b. Children can find an area of a figure given its side lengths, but can also create a rectangle that has a given area. 7c. Demonstrate 3 x 6 = (3 x 2) + (3 x 4) with tiles in a rectangular array  7d. none needed	5. none needed 6. none needed 7. none needed	3. a./b. none needed 4. none needed 5a. Determine the volume of the prism based on the number of cubes in the bottom layer. Expect responses such as "adding the same number of cubes in each layer as were on the bottom layer" or "multiply the number of cubes in one layer times the number of layers". 5b. Students should work to derive the formula for volume, not just apply a given rule. 5c. none needed
Expectations for Learning (Benchmark Indicators)			
Program Correlations			

Domain	**Measurement and Data** **Grade 3**	**Measurement and Data** **Grade 4**	**Measurement and Data** **Grade 5**
Cluster	***Geometric measurement: understand concepts of area and relate area to multiplication and to addition.***	***Geometric measurement: understand concepts of angle and measure angles.***	***Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.***
Investigations	**Standard 5:** Unit 4: *Perimeter, Angles, and Area:2-D Geometry and Measurement* **Standard 6:** Unit 4: *Perimeter, Angles, and Area:2-D Geometry and Measurement* **Standard 7:** Unit 4: *Perimeter, Angles, and Area:2-D Geometry and Measurement* Unit 5: *Equal Groups: Multiplication and Division*	**Standard 5:** Unit 4: *Size, Shape, and Symmetry* **Standard 6:** Unit 4: *Size, Shape, and Symmetry* **Standard 7:** Unit 4: *Size, Shape, and Symmetry*	**Standard 3:** Unit 2: *Prisms and Pyramids* **Standard 4:** Unit 2: *Prisms and Pyramids* **Standard 5:** Unit 2: *Prisms and Pyramids*
Everyday Math	**Standard 5:** *3.6, 3.7, 3.8, 9.3, 9.4, 9.10, 9.11, 9.12* **Standard 6:** *3.6, 3.7, 3.8, 4.8* **Standard 7:** *3.7, 3.8, 4.2, 4.8, 4.9, 6.8, 9.3, 9.4,9.10, 9.11, 9.12, 9.13*	**Standard 5**: Unit 6: *Division; Maps; and Measure of Angles* **Standard 6:** Unit 6: *Division; Maps; and Measure of Angles* **Standard 7:** Unit 6: *Division; Maps; and Measure of Angles* Unit 7: *Fractions and Their Use; Probability*	**Standard 3:** Unit 9: *Coordinates, Area, Volume, Capacity* **Standard 4:** Unit 9: *Coordinates, Area, Volume, Capacity* **Standard 5:** Unit 9: *Coordinates, Area, Volume, Capacity* Unit 11: *Volume*

Domain	**Measurement and Data Grade 3**	**Measurement and Data Grade 4**	**Measurement and Data Grade 5**
Cluster	***Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.***		
Standards	**3.MD.8.** Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.		
Content Elaborations	8. Students should use models, grids, and diagrams to develop an understanding of the concept.		
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Investigations	**Standard 8**: Unit 4: *Perimeter, Angles, and Area*		
Everyday Math	**Standard 8:** *3.4, 3.6, 3.8, 6.5, 6.6, 6.8, 9.3*		

Domain	**Statistics and Probability Grade 6**	**Statistics and Probability Grade 7**	**Statistics and Probability Grade 8**
Cluster	***Develop understanding of statistical variability.***	***Use random sampling to draw inferences about a population.***	***Investigate patterns of association in bi-variate data.***
Standards	**6.SP.1.** Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers. For example, "How old am I?" is not a statistical question, but "How old are the students in my school?" is a statistical question because one anticipates variability in students' ages. **6.SP.2.** Understand that a set of data collected to answer a statistical question has a distribution which can be described by its center, spread, and overall shape. **6.SP.3.** Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.	**7.SP.1.** Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences. **7.SP.2.** Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions. For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be.	**8.SP.1.** Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative association, linear association, and nonlinear association. **8.SP.2.** Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line. **8.SP.3.** Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept. For example, in a linear model for a biology experiment, interpret a slope of 1.5 cm/hr as meaning that an additional hour of sunlight each day is associated with an additional 1.5 cm in mature plant height. **8.SP.4.** Understand that patterns of association can also be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table. Construct and interpret a two-way table summarizing data on two categorical variables collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible association between the two

Domain	**Statistics and Probability Grade 6**	**Statistics and Probability Grade 7**	**Statistics and Probability Grade 8**
Cluster	***Develop understanding of statistical variability.***	***Use random sampling to draw inferences about a population.***	***Investigate patterns of association in bi-variate data.***
			variables. For example, collect data from students in your class on whether or not they have a curfew on school nights and whether or not they have assigned chores at home. Is there evidence that those who have a curfew also tend to have chores?
Content Elaborations	2. Students will use mean, median, mode, range, maximum and minimum to describe a data set and/or to answer statistical questions. Appropriate representations will allow the students to interpret the overall shape. 3. Recognize that a measure of center *(mean, median)* for a numerical data set summarizes all of its values with a single number, while a measure of variation *(range)* describes how its values vary with a single number. Mode is also used to determine central tendency but can also show variation. General: The important purpose of the number is not the value itself, but the interpretation it provides for the variation of the data. Interpreting different measures of center for the same data develops the understanding of how each measure sheds a different light on the data. The use of a similarity and difference matrix to compare mean, median, mode and range may facilitate understanding the distinctions of purpose between and among the measures of center and spread. The determination of the measures of center and the process for developing graphical representation is the focus of the cluster "Summarize and describe distributions" in the Statistics and Probability domain for Grade 6.	Providing opportunities for students to use real-life situations from science and social studies shows the purpose for using random sampling to make inferences about a population. Make available to students the tools needed to develop the skills and understandings required to produce a representative sample of the general population. One key element of a representative sample is understanding that a random sampling guarantees that each element of the population has an equal opportunity to be selected in the sample. Have students compare the random sample to population, asking questions like "Are all the elements of the entire population represented in the sample?" and "Are the elements represented proportionally?" Students can then continue the process of analysis by determining the measures of center and variability to make inferences about the general population based on the analysis. Provide students with random samples from a population, including the statistical measures. Ask students guiding questions to help them make inferences from the sample.	Students will extend their descriptions and understanding of variation to the graphical displays of bivariate (two variables) data. Scatter plots are the most common form of displaying bivariate data in Grade 8. Provide scatter plots and have students practice informally finding the line of best fit. Students should create and interpret scatter plots, focusing on outliers, positive or negative association, linearity or curvature. By changing the data slightly, students can have a rich discussion about the effects of the change on the graph. Have students use a graphing calculator to determine a linear regression and discuss how this relates to the graph. Students should informally draw a line of best fit for a scatter plot and informally measure the strength of fit. Discussion should include "What does it mean to be above the line, below the line?" The study of the line of best fit ties directly to the algebraic study of slope and intercept. Students should interpret the slope and intercept of the line of best fit in the context of the data. Then students can make predictions based on the line of best fit.

Domain	**Statistics and Probability Grade 6**	**Statistics and Probability Grade 7**	**Statistics and Probability Grade 8**
Cluster	***Develop understanding of statistical variability.***	***Use random sampling to draw inferences about a population.***	***Investigate patterns of association in bi-variate data.***
	Classroom instruction should integrate the two clusters.		
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Connected Math	**Standard 1*:*** Investigation 1, 2, 3 and Unit Project: *Data About Us* **Standard 2*:*** Investigation 1, 2, 3: *Data About Us* **Standard 3:** Investigation 1, 2, 3: *Data About Us*	**Standard 1:** CC Investigation 5: *Variability–Data Distribution* **Standard 2:** CC Investigation 5: *Variability–Data Distribution*	**Standard 1:** Investigation 4: *Samples & Populations* **Standard 2:** Investigation 2: *Thinking with Mathematical Models* **Standard 3:** Investigation 2, 3: *Shapes of Algebra* Investigation 2, 3: *Thinking with Mathematical Models* **Standard 4:** CC Investigation 5: *Variability-Samples & Populations*

Domain	**Statistics and Probability** **Grade 6**	**Statistics and Probability** **Grade 7**	**Statistics and Probability** **Grade 8**
Cluster	***Summarize and describe distributions.***	***Draw informal comparative inferences about two populations.***	
Standards	**6.SP.4.** Display numerical data in plots on a number line, including dot plots, histograms, and box plots. **6.SP.5.** Summarize numerical data sets in relation to their context, such as by: a. Reporting the number of observations. b. Describing the nature of the attribute under investigation, including how it was measured and its units of measurement. c. Giving quantitative measures of center (median and/or mean) and variability (interquartile range and/or mean absolute deviation), as well as describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered. d. Relating the choice of measures of center and variability to the shape of the data distribution and the context in which the data were gathered.	**7.SP.3.** Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability. For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team, about twice the variability (mean absolute deviation) on either team; on a dot plot, the separation between the two distributions of heights is noticeable. **7.SP.4.** Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations. For example, decide whether the words in a chapter of a seventh-grade science book are generally longer than the words in a chapter of a fourth-grade science book.	
Content Elaborations	4. Students need to display the same data using different representations. By comparing the different graphs of the same data, students develop understanding of the benefits of each type of representation. 5. Students can come to conclusions and/or make statements based on the landmarks obtained from the data. General: Continue to have students connect contextual situations to data to describe the data set in words prior to computation. Therefore, determining the measures of spread and measures of center mathematically need to follow the	Provide opportunities for students to deal with small populations, determining measures of center and variability for each population. Then have students compare those measures and make inferences. The use of graphical representations of the same data (Grade 6) provides another method for making comparisons. Students begin to develop understanding of the benefits of each method by analyzing data with both methods. When students study large populations, random sampling is used as a basis for the population inference. This build on the skill developed in the Grade 7 cluster "Use random sampling to draw inferences about a population" of Statistics and	

Domain	**Statistics and Probability Grade 6**	**Statistics and Probability Grade 7**	**Statistics and Probability Grade 8**
Cluster	***Summarize and describe distributions.***	***Draw informal comparative inferences about two populations.***	
	development of the conceptual understanding. Students should experience data which reveals both different and identical values for each of the measures. Students need opportunities to explore how changing a part of the data may change the measures of center and measure of spread. Also, by discussing their findings, students will solidify understanding of the meanings of the measures of center and measures of variability, what each of the measures do and do not tell about a set of data, all leading to a better understanding of their usage.	Probability. Measures of center and variability are used to make inferences on each of the general populations. Then the students have make comparisons for the two populations based on those inferences. Have students investigate how advertising agencies uses data to persuade customers to use their products. Additionally, provide students with two populations and have them use the data to persuade both sides of an argument.	
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Connected Math	**Standard 4:** Investigation 1, 3: *Data About Us* CC Investigation 5: *Data About Us* **Standard 5:** *a.* Investigation 1,2,3,4: *How Likely Is It?* *b.* Investigation 1, 2: *Data About Us* *c.* Investigation 3: *Data About Us* CC Investigations 5: *Data About Us* *d.* Investigation 3: *Data About Us*	**Standard 3:** Investigation 2: *Data Distributions* CC Investigations 5: *Variability-Data Distribution* **Standard 4:** Investigation 3, 4: *Data Distributions*	

Domain	Statistics and Probability Grade 6	Statistics and Probability Grade 7	Statistics and Probability Grade 8
Cluster		***Investigate chance processes and develop, use, and evaluate probability models.***	
Standards		**7.SP.5.** Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around 1/2 indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event. **7.SP.6.** Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times. **7.SP.7.** Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy. a. Develop a uniform probability model by assigning equal probability to all outcomes, and use the model to determine probabilities of events. For example, if a student is selected at random from a class, find the probability that Jane will be selected and the probability that a girl will be selected. b. Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. For example, find the approximate probability that a spinning penny will land heads up or that a tossed paper cup will land open-end down. Do the outcomes for the spinning penny appear to be equally likely based on the observed frequencies?	

		7.SP.8. Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation. a. Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs. b. Represent sample spaces for compound events using methods such as organized lists, tables and tree diagrams. For an event described in everyday language (e.g., "rolling double sixes"), identify the outcomes in the sample space which compose the event. c. Design and use a simulation to generate frequencies for compound events. For example, use random digits as a simulation tool to approximate the answer to the question: If 40% of donors have type A blood, what is the probability that it will take at least 4 donors to find one with type A blood?	
Content Elaborations		5. Provide students with situations that have clearly defined probability of never happening as zero, always happening as 1 or equally likely to happen as to not happen as 1/2. Then advance to situations in which the probability is somewhere between any two of these benchmark values. 6. Think about theoretical versus experimental probability. 7. Have students develop probability models to be used to find the probability of events. Provide students with models of equal outcomes and models of not equal outcomes are developed to be used in determining the probabilities of events. 8. Provide students with experiences that require the use of these graphic organizers to determine the theoretical probabilities. Have them practice making the connections between the process of creating lists, tree diagrams, etc. and the interpretation of those models. General: To help students with the discussion of	

		probability, don't allow the symbol manipulation/conversions to detract from the conversations. By having students use technology such as a graphing calculator or computer software to simulate a situation and graph the results, the focus is on the interpretation of the data. Students then make predictions about the general population based on these probabilities.	
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Connected Math		**Standard 5:** Investigation 1 ACE #14: *What Do You Expect?* **Standard 6:** Investigation 1-4: *What Do You Expect?* **Standard 7:** Investigation 1-4: *What Do You Expect?* **Standard 8:** Investigation 1-4: *What Do You Expect?*	

Domain	**Number and Operations in Base Ten Kindergarten**	**Number and Operations in Base Ten Grade 1**	**Number and Operations in Base Ten Grade 2**
Cluster	***Work with number 11-19 to gain foundations for place value***	***Extend the counting sequence***	***Understand place value***
Standards	**K.NBT.1.** Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.	**1.NBT.1.** Count to 120, starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral.	**2.NBT.1.** Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases: 100 can be thought of as a bundle of ten tens — called a "hundred." The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones). **2.NBT.2.** Count within 1000; skip-count by 5s, 10s, and 100s. **2.NBT.3.** Read and write numbers to 1000 using base-ten numerals, number names, and expanded form. **2.NBT.4.** Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using >, =, and < symbols to record the results of comparisons.
Content Elaborations	1. Please note that the understanding of teen numbers is about **ten ones** plus some more ones, **not** a group of ten and some more ones.		1. This involves three understandings of 100 (100 ones; 10 tens; and 1 hundred) 4. Students understand the symbols and what they represent *For all above standards, use models to develop content understanding
Expectations for Learning (Benchmark Indicators)			

Washington West Supervisory Union
Mathematics Curriculum
K-5 Number and Operations in Base 10

Domain	**Number and Operations in Base Ten Kindergarten**	**Number and Operations in Base Ten Grade 1**	**Number and Operations in Base Ten Grade 2**
Cluster	***Work with number 11-19 to gain foundations for place value***	***Extend the counting sequence***	***Understand place value***
Program Correlations			
Investigations	**Standard 5:** Unit 6: *How Many Do You Have?*	**Standard 1:** Unit 1: *How Many of Each* Unit 2: *Making Shapes and Designing Quilts* Unit 3: *Solving Story Problems* Unit 4: *What Would You Rather Be?* Unit 5: *Fish Lengths and Animal Lengths* Unit 6: *Number Games and Crayon Puzzles* Unit 7: *Color, Shape, and Number Puzzles* Unit 8: *Twos, Fives, Tens*	**Standard 1:** Unit 6: *How Many Tens? How Many Ones?* Unit 8: *Partners,Teams and Paper Clips* **Standard 2:** Unit 1: *Counting Coins and Combinations* Unit 2: *Shapes, Blocks and Symmetry* Unit 3: *Stickers, Number Strings, and Story Problems* Unit 4: *Pockets, Teeth and Favorite Things* Unit 5: *How Many Floors? How Many Rooms?* Unit 6: *How Many Tens? How Many Ones?* Unit 7: *Parts of a Whole, Parts of a Group* Unit 8: *Partners, Teams and Paper Clips* **Standard 3:** Unit 1: *Counting Coins and Combinations* Unit 5: *How Many Floors? How Many Rooms?* Unit 6: *How Many Tens? How Many Ones?* **Standard 4:** Unit 6: *How Many Tens? How Many Ones?*
Everyday Math		**Standard 1:** Unit 1: *Establishing Routines* Unit 2: *Everyday Use of Numbers* Unit 4: *Measurement and Basic Facts* Unit 5: *Place Value, Number Stories and Basic Facts* Unit 9: *Place Value and Fractions*	

Domain	Number and Operations in Base Ten Kindergarten	Number and Operations in Base Ten Grade 1	Number and Operations in Base Ten Grade 2
Cluster		***Understand place value***	
Standards		**1.NBT.2.** Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases: a. 10 can be thought of as a bundle of ten ones — called a "ten." b. The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones. c. The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones). **1.NBT.3.** Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols >, =, and <.	
Content Elaborations		2b. Please note shift from kindergarten in the understanding of teen numbers as ten ones and some more ones to one group of ten and some more	
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Investigations		**Standard 2:** Unit 6: *Number Games and Crayon Puzzles* Unit 8: *Twos, Fives, Tens* **Standard 3:** Unit 1: *How Many of Each* Unit 3: *Solving Story Problems* Unit 4: *What Would You Rather Be?* Unit 5: *Fish Lengths and Animal Lengths* Unit 6: *Number Games and Crayon Puzzles* Unit 7: *Color, Shape, and Number Puzzles* Unit 8: *Twos, Fives, Tens* Unit 9: *Blocks and Boxes*	

Domain	**Number and Operations in Base Ten** **Kindergarten**	**Number and Operations in Base Ten** **Grade 1**	**Number and Operations in Base Ten** **Grade 2**
Cluster		***Understand place value***	
Everyday Math		**Standard 2:** Unit 5: *Place Value, Number Stories and Math Facts* Unit 8: *Mental Arithmetic, Money and Fractions* Unit 10: *Year End Review* **Standard 3:** Unit 1: *Establishing Routines* Unit 5: *Place Value, Number Stories and Math Facts*	

Domain	Number and Operations in Base Ten Kindergarten	Number and Operations in Base Ten Grade 1	Number and Operations in Base Ten Grade 2
Cluster		***Use place value understanding and properties of operations to add and subtract.***	***Use place value understanding and properties of operations to add and subtract***
Standards		**1.NBT.4.** Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten. **1.NBT.5.** Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used. **1.NBT.6.** Subtract multiples of 10 in the range 10-90 from multiples of 10 in the range 10-90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.	**2.NBT.5.** Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction. **2.NBT.6.** Add up to four two-digit numbers using strategies based on place value and properties of operations. **2.NBT.7.** Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds. **2.NBT.8.** Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900. **2.NBT.9.** Explain why addition and subtraction strategies work, using place value and the properties of operations.
Content Elaborations		4. Students are expected to use models and materials to do this work; and not expected to work with numbers only. 6. Students are expected to use models and materials to do this work; and not expected to work with numbers only.	7. The use of models, materials, including diagrams is expected to be used when performing operations with these numbers. 9. Students should able to develop and defend their strategies based on a thorough understanding of the actions of addition and subtraction.

Domain	**Number and Operations in Base Ten Kindergarten**	**Number and Operations in Base Ten Grade 1**	**Number and Operations in Base Ten Grade 2**
Cluster		***Use place value understanding and properties of operations to add and subtract.***	***Use place value understanding and properties of operations to add and subtract***
			9. It is vital that student-invented strategies be shared, explored, recorded, and tried by others to develop flexibility of strategies for addition and subtraction.
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Investigations		**Standard 4:** Unit 8: *Twos, Fives, Tens* **Standard 5:** Unit 8: *Twos, Fives, Tens* **Standard 6:** Unit 8: *Twos, Fives, Tens*	**Standard 5:** Unit 1: *Counting Coins and Combinations* Unit 3: *Stickers, Number Strings, and Story Problems* Unit 4: *Pockets, Teeth and Favorite Things* Unit 5: *How Many Floors? How Many Rooms?* Unit 6: *How Many Tens? How Many Ones?* Unit 7: *Parts of a Whole, Parts of a Group* Unit 8: *Partners, Teams and Paper Clips* Unit 9: *Measuring Length and Time* **Standard 6:** Unit 3: *Stickers, Number Strings, and Story Problems* Unit 5: *How Many Floors? How Many Rooms?* Unit 6: *How Many Tens? How Many Ones?* Unit 8: *Partners, Teams and Paper Clips* **Standard 7:** Unit 1: *Counting Coins and Combinations* Unit 8: *Partners, Teams and Paper Clips* **Standard 8:** Unit 6: *How Many Tens? How Many Ones?* **Standard 9:** Unit 1: *Counting Coins and Combinations* Unit 3: *Stickers, Number Strings, and Story Problems* Unit 6: *How Many Tens? How Many Ones?* Unit 8: *Partners, Teams and Paper Clips*

Domain	**Number and Operations in Base Ten** **Kindergarten**	**Number and Operations in Base Ten** **Grade 1**	**Number and Operations in Base Ten** **Grade 2**
Cluster		***Use place value understanding and properties of operations to add and subtract.***	***Use place value understanding and properties of operations to add and subtract***
Everyday Math		**Standard 4:** Unit 2: *Everyday Use of Math* Unit 5: *Place Value, Number Stories, and Basic Facts* Unit 8: *Mental Arithmetic, Money, and Fractions* Unit 9: *Place Value and Arithmetics* Unit 10: *Year End Review* **Standard 5:** Unit 9: *Place Value and Arithmetics* Unit 10: *Year End Review*	

Domain	Number and Operations in Base Ten Grade 3	Number and Operations in Base Ten Grade 4	Number and Operations in Base Ten Grade 5
Cluster	***Use place value understanding and properties of operations to perform multi-digit arithmetic***	***Generalize place value understanding for multi-digit whole numbers***	***Understand the place value system***
Standards	**3.NBT.1.** Use place value understanding to round whole numbers to the nearest 10 or 100. **3.NBT.2.** Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction. **3.NBT.3.** Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations.	**4.NBT.1.** Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. *For example, recognize that 700 ÷ 70 = 10 by applying concepts of place value and division.* **4.NBT.2.** Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons. **4.NBT.3.** Use place value understanding to round multi-digit whole numbers to any place.	**5.NBT.1.** Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left. **5.NBT.2.** Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10. **5.NBT.3.** Read, write, and compare decimals to thousandths. Read and write decimals to thousandths using base-ten numerals, number names, and expanded form, e.g., 347.392 = 3 × 100 + 4 × 10 + 7 × 1 + 3 × (1/10) + 9 × (1/100) + 2 × (1/1000). Compare two decimals to thousandths based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons. **5.NBT.4.** Use place value understanding to round decimals to any place.

Domain	Number and Operations in Base Ten Grade 3	Number and Operations in Base Ten Grade 4	Number and Operations in Base Ten Grade 5
Cluster	***Use place value understanding and properties of operations to perform multi-digit arithmetic***	***Generalize place value understanding for multi-digit whole numbers***	***Understand the place value system***
Content Elaborations	3. This standard was formally in 4th grade and involves building understanding with models - not just sticking a zero on the end	2. Expanded form is expected as a way to notate a number (for example 2,375 = 2,000 +300+70+5). 2. Students should use understanding of place value to compare numbers.	2. It is important to use models and materials to show the powers of ten; this is about understanding the structure of the number, and not just about changing the position of a decimal point or "putting on more zeros."
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Investigations	**Standard 1**: Unit 3: *Collections and Travel Stories* Unit 4: *Perimeter, Angles, and Area* Unit 6: *Stories, Tables, and Graphs* Unit 7: *Finding Fair Shares* Unit 9: *Solids and Boxes* **Standard 2**: Unit 1: *Trading Stickers, Combining Coins* Unit 3: *Collections and Travel Stories* Unit 4: *Perimeter, Angles, and Area* Unit 6: *Stories, Tables, and Graphs* Unit 7: *Finding Fair Shares* Unit 8: *How Many Hundreds? How Many Miles?* Unit 9: *Solids and Boxes* **Standard 3**: Unit 5: *Equal Groups*	**Standard 1:** Unit 5: *Landmarks and Large Numbers* **Standard 2:** Unit 5: *Landmarks and Large Numbers* Unit 6: *Fraction Cards and Decimal Squares* Unit 7: *Moving Between Solids and Silhouettes* **Standard 3:** Unit 5: *Landmarks and Large Numbers*	**Standard 1:** Unit 3: *Thousands of Miles, Thousands of Seats* Unit 6: *Decimals on Grids and Number Lines* **Standard 2:** Unit 1: *Number Puzzles and Multiple Towers* Unit 6: *Decimals on Grids and Number Lines* **Standard 3:** Unit 6: *Decimals on Grids and Number Lines* Unit 8: *Growth Patterns* **Standard 4:** Unit 6: *Decimals on Grids and Number Lines*
Everyday Math	**Standard 1:** Unit: *1.11, 2.7, 2.8, 7.7, 9.5* **Standard 2:** Unit: *1.8, 1.9, 1.10, 1.11, 2.1, 2.2, 2.3, 2.4, 2.7, 2.8, 2.9, 9.3, 9.4, 9.5, 9.9, 9.11, 9.12, 9.13*	**Standard 1**: Unit 2: *Using Numbers and Organizing Data* Unit 4: *Decimals and Their Use* Unit 5*: Big Numbers, Estimation, and Computation* **Standard 2**: Unit 2: *Using Numbers and Organizing Data*	**Standard 1:** Unit 2: *Estimation and Computations* Unit 7: *Exponents and Negative Numbers* **Standard 2:** Unit 1: *Number Theory* Unit 2: *Estimation and Computation* Unit 3: *Geometry Exploration and AT*

Domain	Number and Operations in Base Ten Grade 3	Number and Operations in Base Ten Grade 4	Number and Operations in Base Ten Grade 5
Cluster	***Use place value understanding and properties of operations to perform multi-digit arithmetic***	***Generalize place value understanding for multi-digit whole numbers***	***Understand the place value system***
	Standard 3: Unit: 7.6, 7.7, 7.8, 9.1, 9.2, 9.3, 9.11, 9.12	Unit 5: *Big Numbers, Estimation, and Computation* **Standard 3:** Unit 5*: Big Numbers, Estimation, and Computation*	Unit 4: *Division* Unit 7: *Exponents and Negative Numbers* Unit 9: *Coor, Area, Volume and Capacity* Unit 10*: Using Data and Algebraic Concepts and Skills* Unit 11: *Volume* **Standard 3:** Unit 2: *Estimation and Computation* Unit 3*: Geometry Exploration and AT* Unit 5: *Fractions, Decimals, Percents* Unit 7: *Exponents and Negative Numbers* **Standard 4:** Unit 2: *Estimation and Computations* Unit 3*: Geometry Exploration and AT* Unit 5: *Fractions, Decimals, Percents* Unit 10: *Using Data and Algebraic Concepts and Skills* Unit 11: *Volume*

Domain	**Number and Operations in Base Ten Grade 3**	**Number and Operations in Base Ten Grade 4**	**Number and Operations in Base Ten Grade 5**
Cluster		***Use place value understanding and properties of operations to perform multi-digit arithmetic***	***Perform operations with multi-digit whole numbers and decimals to hundredths***
Standards		**4.NBT.4.** Fluently add and subtract multi-digit whole numbers using the standard algorithm. **4.NBT.5.** Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. **4.NBT.6.** Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.	**5.NBT.5.** Fluently multiply multi-digit whole numbers using the standard algorithm. **5.NBT.6.** Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. **5.NBT.7.** Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.
Content Elaborations		4. Students are expected to use the standard algorithm fluently, though it should not be their sole strategy. It is important to develop understanding of standard algorithm through the use of models and a thorough understanding of number structure.	5. This must be developed through the use of models and materials so that the standard algorithm makes sense as a solution strategy; students are able to use models and materials to defend the solution arrived at using the standard algorithm.
Expectations for Learning (Benchmark Indicators)			

Domain	**Number and Operations in Base Ten Grade 3**	**Number and Operations in Base Ten Grade 4**	**Number and Operations in Base Ten Grade 5**
Cluster		***Use place value understanding and properties of operations to perform multi-digit arithmetic***	***Perform operations with multi-digit whole numbers and decimals to hundredths***
Program Correlations			
Investigations		**Standard 4:** Unit 2: *Describing the Shape of the Data* Unit 4: *Size, Shape, Shape and Symmetry* Unit 5: *Landmarks and Large Numbers* **Standard 5:** Unit 3: *Multiple Towers and Division Stories* Unit 8**:** *How Many Packages? How Many Groups?* Unit 9: *Penny Jars and Plant Growth* **Standard 6:** Unit 3: *Multiple Towers and Division Stories* Unit 8: *How Many Packages? How Many Groups?* Unit 9: *Penny Jars and Plant Growth*	**Standard 5:** Unit 1: *Number Puzzles and Multiple Towers* Unit 2: *Prisms and Pyramids* Unit 3*: Thousands of Miles, Thousands of Seats* Unit 6: *Decimals on Grids and Number Lines* Unit 7: *How Many People? How Many Teams?* Unit 9*: How Long Can You Stand On One Foot* **Standard 6:** Unit 1*: Number Puzzles and Multiple Towers* Unit 2: *Prisms and Pyramids* Unit 3*: Thousands of Miles, Thousands of Seats* Unit 6: *Decimals on Grids and Number Lines* Unit 7*: How Many People? How Many Teams?* Unit 9*: How Long Can You Stand On One Foot* **Standard 7:** Unit 6*: Decimals on Grids and Number Lines*
Everyday Math		**Standard 4**: Unit 2: *Using Numbers and Organizing Data* **Standard 5**: Unit 5*: Big Numbers, Estimation, and Computation* **Standard 6**: Unit 6: *Division; Map Ref: Measures of Angles*	**Standard 5:** Unit 2: *Estimation and Computation* **Standard 6:** Unit 4: *Division* **Standard 7**: Unit 2: *Estimation and Computation* Unit 4: *Division*

Domain	**The Number System Grade 6**	**The Number System Grade 7**	**The Number System Grade 8**
Cluster	***Apply and extend previous understandings of multiplication and division to divide fractions by fractions.***	***Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.***	***Know that there are numbers that are not rational, and approximate them by rational numbers.***
Standards	**6.NS.1.** Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for (2/3) ÷ (3/4) and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that (2/3) ÷ (3/4) = 8/9 because 3/4 of 8/9 is 2/3. (In general, (a/b) ÷ (c/d) = ad/bc.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?	**7.NS.1.** Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram. a. Describe situations in which opposite quantities combine to make 0. For example, a hydrogen atom has 0 charge because its two constituents are oppositely charged. b. Understand p + q as the number located a distance |q| from p, in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts. c. Understand subtraction of rational numbers as adding the additive inverse, p – q = p + (–q). Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts. d. Apply properties of operations as strategies to add and subtract rational numbers. **7.NS.2.** Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers. a. Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such	**8.NS.1.** Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number. **8.NS.2.** Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π2). For example, by truncating the decimal expansion of √2, show that √2 is between 1and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.

		as (−1)(−1) = 1 and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts. b. Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then −(p/q) = (−p)/q = p/(−q). Interpret quotients of rational numbers by describing real-world contexts. c. Apply properties of operations as strategies to multiply and divide rational numbers. d. Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats. **7.NS.3**. Solve real-world and mathematical problems involving the four operations with rational numbers.	
Content Elaborations	1. Computation with fractions is best understood when it builds upon the familiar understandings of whole numbers (12 ÷ 3 and 4 ÷ ½) and is paired with visual representations (area or linear models). Solve a simpler problem with whole numbers, and then use the same steps to solve a fraction divided by a fraction. Looking at the problem through the lens of "How many groups?" or "How many in each group?" helps visualize what is being sought. Link the division of fractions to a missing factor multiplication problem (½ ÷ ¼ seen as ¼ x ___ = ½) and connect to the inverse relationship between multiplication and division. Learning how to compute fraction division problems is one part, being able to relate the problems to real-world situations is important. Providing opportunities to	1-3. Learning moves to exploring and ultimately formalizing rules for operations (addition, subtraction, multiplication and division) with integers. 1-3. This standard is not about teaching the "rules" (a negative times a negative equals a positive). A conceptual understanding of all operations with rational numbers is important before students use developed "rules". 2d. Providing examples with families of fractions, such as, sevenths, ninths, thirds, etc. to convert to decimals using long division and grouping the equivalent as terminating or repeating are helpful for students to begin to see why these patterns occur.	1. In Grade 7 students convert a rational number to a decimal using long division, they have not yet developed strategies to do the inverse (decimal to fraction). 1 and 2. In the process of distinguishing between rational and irrational numbers, some students are surprised that the decimal representation of pi does not repeat. Some students believe that if only we keep looking at digits farther and farther to the right, eventually a pattern will emerge.

	create stories for fraction problems or writing equations for situations is needed. Students may believe that dividing by ½ is the same as dividing in half. Dividing by half means to find how many ½s there are in a quantity, whereas, dividing in half means to take a quantity and split it into two equal parts. Thus 7 divided by ½ = 14 and 7 divided in half = 3 ½ Students need to be clear on which fraction represents the dividend and the divisor (so in ½ ÷ ¼, ½ is the dividend and ¼ is the divisor). The problem is looking for how many ¼s are in ½? Teaching "invert and multiply" without developing an understanding of why it works first leads to confusion as to when to apply the shortcut.		
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Connected Math	**Standard 1:** Investigation 4: *Bits and Pieces II*	**Standard 1a:** Investigation 2: *Accentuate the Negative* **Standard 1b:** Investigation 1-2: *Accentuate the Negative* **Standard 1c:** Investigation 2: *Accentuate the Negative* **Standard 1d:** Investigation 2, 4: *Accentuate the Negative* **Standard 2a:** Investigation 3, 4: *Accentuate the Negative* **Standard 2b:** Investigation 3: *Accentuate the Negative* **Standard 2c:** Investigation 3, 4: *Accentuate the Negative*	**Standard 1:** Investigation 4: *Looking for Pythagoras* **Standard 2:** Investigation 4: *Looking for Pythagoras*

		Standard 2d: Investigation 3: *Comparing and Scaling* Investigation 3: *Accentuate the Negative* **Standard 3:** Investigation 2-4: *Accentuate the Negative*	

Domain	**The Number System Grade 6**	**The Number System Grade 7**	**The Number System Grade 8**
Cluster	***Compute fluently with multi-digit numbers and find common factors and multiples.***		
Standards	**6.NS.2.** Fluently divide multi-digit numbers using the standard algorithm. **6.NS.3.** Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation. **6.NS.4.** Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor. For example, express 36 + 8 as 4 (9 + 2).		
Content Elaborations	2. As students study whole numbers in the elementary grades, a foundation is laid in the conceptual understanding of each operation. Discovering and applying multiple strategies for computing creates connections which evolve into the proficient use of standard algorithms. Fluency with an algorithm denotes an ability that is efficient, accurate, appropriate and flexible. 3. It is important to develop the conceptual understanding of the movement of the decimal point when multiplying and dividing as more than just the rule of counting the places and moving the decimal point.		

	Example 0.02 x 0.9 = 0.018 as (2 x 1/100) x (9 x 1/10) = (2x9) x (1/100 x 1/10) = (2x9) x (1/1000) 4. Greatest common factor and least common multiple are usually taught as a means of combining fractions with unlike denominators. This cluster builds upon the previous learning of the multiplicative structure of whole numbers, as well as prime and composite numbers in Grade 4. Although the process is the same, the point is to become aware of the relationships between numbers and their multiples. For example, consider answering the question: "If two numbers are multiples of four, will the sum of the two numbers also be a multiple of four?" Being able to see and write the relationships between numbers will be beneficial as further algebraic understandings are developed. Another focus is to be able to see how the GCF is useful in expressing the numbers using the distributive property, (36 + 24) = 12(3+2), where 12 is the GCF of 36 and 24.		
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Connected Math	**Standard 2:** Investigation 3: *Bits and Pieces I* Investigation 3: *Bits and Pieces III* Investigation 2: *Shapes and Designs*		

	Standard 3: Investigation 1-3: *Bits and Pieces III* **Standard 4:** Investigation 1, 2: *Prime Time* CC Investigation: *Number Properties and Algebraic Expressions - Bits and Pieces III*		

Domain	**The Number System Grade 6**	**The Number System Grade 7**	**The Number System Grade 8**
Cluster	***Apply and extend previous understandings of numbers to the system of rational numbers.***		
Standards	**6.NS.5**. Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation. **6.NS.6.** Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates. a. Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., – (–3) = 3, and that 0 is its own opposite. b. Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes. c. Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a		

	coordinate plane. **6.NS.7.** Understand ordering and absolute value of rational numbers. a. Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram. For example, interpret –3 > –7 as a statement that –3 is located to the right of –7 on a number line oriented from left to right. b. Write, interpret, and explain statements of order for rational numbers in real-world contexts. For example, write –3ºC > –7ºC to express the fact that –3ºC is warmer than –7ºC. c. Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. For example, for an account balance of –30 dollars, write |–30| = 30 to describe the size of the debt in dollars. d. Distinguish comparisons of absolute value from statements about order. For example, recognize that an account balance less than –30 dollars represents a debt greater than 30 dollars. **6.NS.8.** Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.		
Content Elaborations	5 and 6. Demonstration of understanding of positives and negatives involves translating among words, numbers and models: given the words "7 degrees below zero," showing it on a thermometer and writing -7; given -4 on a number line, writing a real-life example and mathematically -4. Number lines also		

	give the opportunity to model absolute value as the distance from zero. 5 and 6. Using number lines to model negative numbers, prove the distance between opposites, and understand the meaning of absolute value easily transfers to the creation and usage of four-quadrant coordinate grids. Points can now be plotted in all four quadrants of a coordinate grid. Differences between numbers can be found by counting the distance between numbers on the grid. Actual computation with negatives and positives is handled in Grade 7. 5 and 6. The misconception that -11 should be placed above -10 on a vertical number line needs to be addressed. Exposure to the translation of a horizontal number line to a vertical number line is important to help clarify. 7. 9 > 3 so students may assume -9 > -3		
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Connected Math	**Standard 5:** Investigation 2: *Bits and Pieces II* **Standard 6:** Investigation 1-4: *Bits and Pieces I,II,III*		

	Standard 6a*:* Investigation 2 ACE #51: *Bits and Pieces II* CC Investigation 3: *Integers and the Coordinate Plane* **Standard 6b:** CC Investigation 3: *Integers and the Coordinate Plane* **Standard 6c:** CC Investigation 3: *Integers and the Coordinate Plane* **Standard 7a:** Investigation 1-4: *Bits and Pieces I* Investigation 2 ACE #51: *Bits and Pieces II* **Standard 7b:** Investigation 2 ACE #51: *Bits and Pieces II* Investigation 1 ACE #58*: Bits and Pieces III* **Standard 7c:** CC Investigation 3: *Integers and the Coordinate Plane – Bits and Pieces III* **Standard 7d:** CC Investigation 3: *Integers and the Coordinate Plane - Bits & Pieces III* **Standard 8:** Investigation 2: *Covering and Surrounding* Investigation 2: *Data About Us* CC Investigations 3: *Integers and the Coordinate Plane - Bits & Pieces III*		

Domain	Operations and Algebraic Thinking Kindergarten	Operations and Algebraic Thinking Grade 1	Operations and Algebraic Thinking Grade 2
Cluster	***Understand addition as putting together and adding to, and understand subtraction***	***Represent and solve problems involving addition and subtraction.***	***Represent and solve problems involving addition and subtraction.***
Standards	**K.OAT.1.** Represent addition and subtraction with objects, fingers, mental images, drawings, sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations. **K.OAT.2.** Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem. **K.OAT.3.** Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1). **K.OAT.4.** For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation. **K.OAT.5**. Fluently add and subtract within 5.	**1.OAT.1.** Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem. **1.OAT.2**. Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.	**2.OAT.1.** Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.
Content Elaborations	This standard is about understanding what **actions** are additive, etc.	1. For example, given a word problem such as "15 children were playing on the playground. Some more children came. Now there are 20 children", children could write that situation as "15 + ___ = 20". Moreover, they may also represent an equation that expresses the way they solved it; for example 20-__=15; 5 children came" or "20 = __ + 15, so 5 must be the number	1. This is predicated on deep understanding of operations; for example, that subtraction is about finding a difference--subtraction thus includes removal, but is not a synonym for it--also includes comparison and missing part situations (as these both are about finding a difference). Students should be able to represent problems

		of children who came to the group”. This standard is predicated on students understanding the actions of the operations of addition and subtraction--that they understand, for example, that subtraction is about finding the difference, and that may be a removal situation, a comparative one, etc.	and solutions with both models and equations. For example, given a word problem such as “67 children were playing on the playground. Some more children came. Now there are 142 children”, students could write that situation as “67 + ___ = 142”. Moreover, they may also represent an equation that expresses the way they solved it; for example 142-__= 67; 75 children came” or “142 = __ + 67, so 75 must be the number of children who came to the group”. Similarly, they should be able to justify those equations with models, and represent solutions using models with equations.
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Investigations	**Standard 1:** Unit 4: *Measuring and Counting* Unit 6: *How Many Do You Have?* **Standard 2:** Unit 4: *Measuring and Counting* Unit 6: *How Many Do You Have?* **Standard 3:** Unit 4: *Measuring and Counting* Unit 6: *How Many Do You Have?* **Standard 4:** Unit 4: *Measuring and Counting* Unit 6: *How Many Do You Have?* **Standard 5:** Unit 6: *How Many Do You Have?*	**Standard 1:** Unit 1: *How Many of Each* Unit 3: *Solving Story Problems* Unit 5: *Fish Lengths and Animal Lengths* Unit 6: *Number Games and Crayon Puzzles* Unit 7: *Color, Shape, and Number Puzzles* Unit 8: *Twos, Fives, Tens* Unit 9: *Blocks and Boxes* **Standard 2:** Unit 1: *How Many of Each* Unit 3: *Solving Story Problems* Unit 6: *Number Games and Crayon Puzzles* Unit 7: *Color, Shape, and Number Puzzles* Unit 8: *Twos, Fives, Tens* Unit 9: *Blocks and Boxes*	**Standard 1:** Unit 1: *Counting Coins and Combinations* Unit 2: S*hapes, Blocks and Symmetry* Unit 3: S*tickers, Number Strings, and Story Problems* Unit 5: *How Many Floors? How Many Rooms?* Unit 8: *Partners, Teams and Paper Clips*
Everyday Math		**Standard 1:** *1.5, 1.1, 2.6, 2.7, 2.8, 2.11, 2.12, 2.13, 3.6, 3.11, 3.12, 3.13, 3.14, 4.3, 4.6, 4.7, 4.9, 5.5,*	**Standard 1:** *2.1, 2.2, 2.6, 2.11, 3.6, 3.7, 3.8, 4.1, 4.2, 4.4, 4.6, 5.2, 6.2, 6.4, 10.3, 10.4, 10.6*

		5.6, 5.7, 5.8, 5.10, 6.3, 6.9, 6.10, 10.3 , 10.4 **Standard 2:** *2.13, 3.10, 8.4*	
Connections			

Domain	**Operations and Algebraic Thinking** **Kindergarten**	**Operations and Algebraic Thinking** **Grade 1**	**Operations and Algebraic Thinking** **Grade 2**
Cluster		*Understand and apply properties of operations and the relationship between addition and subtraction.*	
Standards		**1.OAT.3.** Apply properties of operations as strategies to add and subtract. *Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known (commutative property of addition).To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12 (associative property of addition).* **1.OAT.4.** Understand subtraction as an unknown-addend problem. *For example, subtract 10 – 8 by finding the number that makes 10 when added to 8.*	
Content Elaborations		4. 10 - 8 means find the **difference** between 10 and 8. So a child might add 2 to 8 to get to 10 to calculate that difference, for example.	
Expectations for Learning (Benchmark Indicators)			
Program Correlations:			
Investigations		**Standard 3:** Unit 1: *How Many of Each* Unit 3: *Solving Story Problems* Unit 6: *Number Games and Crayon Puzzles* Unit 8: *Twos, Fives, Tens* **Standard 4 :** Unit 1: *How Many of Each* Unit 3: *Solving Story Problems* Unit 6: *Number Games and Crayon Puzzles*	

Everyday Math		**Standard 3:** *2.13, 3.10, 4.11, 4.12, 5.5, 5.8, 5.11, 6.1, 6.3, 6.4* **Standard 4:** *2.12, 4.11, 5.7, 5.8, 6.3, 6.5, 8.5*	
Connections			

Domain	**Operations and Algebraic Thinking Kindergarten**	**Operations and Algebraic Thinking Grade 1**	**Operations and Algebraic Thinking Grade 2**
Cluster		***Add and subtract within 20.***	***Add and subtract within 20***
Standards		**1.OAT.5**. Relate counting to addition and subtraction (e.g., by counting on 2 to add 2). **1.OAT.6**. Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 – 4 = 13 – 3 – 1 = 10 – 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 – 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13).	**2.OAT.2.** Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers.
Content Elaborations		5. This is about relating the actions of counting on and counting back to actions of addition and subtraction (number line, counting chart very useful for this, as well as fingers and manipulatives). 6. Fluency with facts through 10 (oral and written).	2. Fluency with facts through 20 (oral and written). Note: Written fluency may need to be supplemented.
Expectations for Learning (Benchmark Indicators)			
Program Correlations:			
Investigations		**Standard 5:** Unit 1: *How Many of Each*	**Standard 2:** Unit 1: *Counting Coins and Combinations*

		Unit 3: *Solving Story Problems* Unit 6: *Number Games and Crayon Puzzles* Unit 7: *Color, Shape, and Number Puzzles* Unit 8: *Twos, Fives, Tens* **Standard 6:** Unit 1: *How Many of Each* Unit 3: *Solving Story Problems* Unit 6: *Number Games and Crayon Puzzles* Unit 7: *Color, Shape, and Number Puzzles* Unit 8: *Twos, Fives, Tens* Unit 9: *Blocks and Boxes*	Unit 2: *Shapes, Blocks and Symmetry* Unit 3: *Stickers, Number Strings, and Story Problems* Unit 4: *Pockets, Teeth and Favorite Things* Unit 5: *How Many Floors? How Many Rooms?* Unit 6: *How Many Tens? How Many Ones?* Unit 8: *Partners, Teams and Paper Clips* Unit 9: *Measuring Length and Time*
Everyday Math		**Standard 5:** *2.1, 2.11, 2.13, 3.6, 3.8, 3.9, 3.10. 6.3, 6.8* **Standard 6:** *1.5, 1.10, 1.13, 2.1, 2.2, 2.3, 2.8, 2.11, 2.12, 2.13, 3.6, 3.9, 3.14 , 4.2, 4.6, 4.7, 4.8, 4.11, 4.12, 5.5, 5.7, 5.9, 5.10, 5.11, 5.12, 5.13, 6.1, 6.2, 6.3, 6.4, 6.5, 6.6, 6.7, 6.8, 7.1, 7.2, 7.3, 7.7, 8.2, 8.3, 8.5, 8.7, 8.8, 8.9, 9.1, 9.7*	**Standard 2:** *1.4, 1.8, 1.11, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.11, 2.12, 2.13, 4.3, 4.5, 4.7, 5.2, 5.5, 5.7, 8.1, 8.2, 8.7, 9.2, 9.3, 9.6, 9.9. 12.2, 12.5*
Connections			

Domain	**Operations and Algebraic Thinking Kindergarten**	**Operations and Algebraic Thinking Grade 1**	**Operations and Algebraic Thinking Grade 2**
Cluster		*Work with addition and subtraction equations.*	
Standards		**1.OAT.7.** Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. *For example, which of the following equations are true and which are false? 6 = 6, 7 = 8 – 1, 5 + 2 = 2 + 5, 4 + 1 = 5 + 2.* **1.OAT.8.** Determine the unknown whole number in an addition or subtraction equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations 8 + ? = 11, 5 = ? – 3, 6 + 6 =?.*	
Content Elaborations		Both of these are predicated on a thorough understanding of the equal sign--be sure the children don't interpret it as "The number that follows the = is the answer." Rather, we want them to understand it as "the same as"--the expression or value on one side of the equal sign is the same as the expression or value on the other side.	
Expectations for Learning (Benchmark Indicators)			
Program Correlations:			
Investigations		**Standard 7:** Unit 1: *How Many of Each* Unit 3: *Solving Story Problems*	

		Unit 6: *Number Games and Crayon Puzzles* Unit 7: *Color, Shape, and Number Puzzles* Unit 8: *Twos, Fives, Tens* **Standard 8:** Unit 1: *How Many of Each* Unit 3: *Solving Story Problems* Unit 6: *Number Games and Crayon Puzzles* Unit 7: *Color, Shape, and Number Puzzles* Unit 8: *Twos, Fives, Tens* Unit 9: *Blocks and Boxes*	
Everyday Math		**Standard 7:** *2.11, 3.6, 4.12, 5.3, 5.10, 6.2, 8.2, 12.5, 3.5, 10.6* **Standard 8:** *3.8, 3.9, 4.1, 4.12, 5.8, 5.10, 5.11, 5.12, 5.13, 6.3, 6.4, 6.5, 6.6, 6.8*	
Connections			

Domain	Operations and Algebraic Thinking Kindergarten	Operations and Algebraic Thinking Grade 1	Operations and Algebraic Thinking Grade 2
Cluster			*Work with equal groups of objects to gain foundations for multiplication.*
Standards			**2.OAT.3.** Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends. **2.OAT.4.** Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.
Content Elaborations			4. This builds a foundation for multiplication.
Expectations for Learning (Benchmark Indicators)			
Program Correlations:			
Investigations			**Standard 3:** Unit 3: *Stickers, Number Strings, and Story Problems* Unit 5: *How Many Floors? How Many Rooms?* Unit 6: *How Many Tens? How Many Ones?* Unit 8: *Partners, Teams and Paper Clips* **Standard 4:** Unit 1: *Counting Coins and Combinations* Unit 2: *Shapes, Blocks and Symmetry*

			Unit 3: *Stickers, Number Strings, and Story Problems* Unit 5: *How Many Floors? How Many Rooms?*
Everyday Math			**Standard 3:** *1.10, 1.12, 2.3, 2.4, 2.5, 2.8* **Standard 4:** *5.4, 6.6, 6.7, 6.8, 6.9, 8.2, 11.4, 11.6, 11.7*
Connections			

Domain	**Operations and Algebraic Thinking Grade 3**	**Operations and Algebraic Thinking Grade 4**	**Operations and Algebraic Thinking Grade 5**
Cluster	***Represent and solve problems involving multiplication and division.***	***Use the four operations with whole numbers to solve problems.***	***Write and interpret numerical expressions***
Standards	**3.OAT.1.** Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. For example, describe a context in which a total number of objects can be expressed as 5 × 7. **3.OAT.2.** Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8. **3.OAT.3.** Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem. **3.OAT.4.** Determine the unknown whole number in a multiplication or division equation relating three whole numbers. For example, determine the unknown number that makes the equation true in each of the equations 8 × ? = 48, 5 = ? ÷ 3, 6 × 6 = ?.	**4.OAT.1.** Interpret a multiplication equation as a comparison, e.g., interpret 35 = 5 × 7 as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations. **4.OAT.2.** Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison. **4.OAT.3.** Solve multi-step word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.	**5.OAT.1.** Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols. **5.OAT.2.** Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them. For example, express the calculation "add 8 and 7, then multiply by 2" as 2 × (8 + 7).Recognize that 3 × (18932 + 921) is three times as large as 18932 + 921, without having to calculate the indicated sum or product.

Content Elaborations	1. The example of 5 x 7 can be interpreted as the total number of objects in 7 groups of 5 objects each as well as the number of objects in 5 groups of 7 objects. 2. This looks at two things, number **of** groups and number **in each** group. That is, 56 divided by 8 can mean 56 ÷ into 8 groups or 56 divided into groups of 8. 3. Can use materials to create models/representations. Again, this is predicated on deep understanding of multiplication and division and the relationship between the two. Students should be able to represent problems and solutions with both models and equations. For example, given a word problem such as "72 children are playing on the playground. They want to make teams of 8 kids. How many groups can they make?", children could write that situation as "72 ÷ 8 = ___". Moreover, they may also represent an equation that expresses the way they solved it; for example 8 x __= 72; so they can make 9 teams" or "72 = __ x 8, so 9 must be the number of teams they can make". Similarly, they should be able to justify those equations with models, and represent solutions using models with equations.	2. This work scaffolds and builds subsequent work in other grades with functions and proportional thinking. Examples of this kind of work at grade 4 may include: "*Helen raised $12 for the food bank last year and she raised 6 times as much money this year. How much money did she raise this year? *Sandra raised $15 for the PTA and Nita raised $45. How many times as much money did Nita raise as compared to Sandra? *Nita raised $45 for the PTA, which was 3 times as much money as Sandra raised. How much money did Sandra raise?" from http://www.illustrativemathematics.org/standards/k8 3. This emphasis is on multi-step problems. Emphasize this standard by using multiple problem solving examples for application.	1. In mathematical order of operations, one solves work in braces, then brackets, and then parentheses. { [2 x (12 + 8)] ÷ 4 } + 3 x (4 x 5) = 70 { } Braces [ ] Brackets ( ) Parentheses
Expectations for Learning (Benchmark Indicators)			

Program Correlations			
Investigations	**Standard 1:** Unit 5: *Equal Groups* **Standard 2:** Unit 5: *Equal Groups* **Standard 3:** Unit 5: *Equal Groups* Unit 6: *Stories, Tables, and Graphs* Unit 7: *Finding Fair Shares* Unit 8: *How Many Hundreds? How Many Miles?* **Standard 4:** Unit 5: *Equal Groups*	**Standard 1:** Unit 1: *Factors, Multiples, and Arrays* Unit 3: *Multiple Towers and Division Stories* **Standard 2:** Unit 1: *Factors, Multiples, and Arrays* Unit 3: *Multiple Towers and Division Stories* **Standard 3:** Unit 1: *Factors, Multiples, and Arrays* Unit 3: *Multiple Towers and Division Stories* Unit 8: *How Many Packages? How Many Groups?*	**Standard 1:** Unit 1: *Number Puzzles and Multiple Towers* Unit 2*: Prisms and Pyramids* Unit 6: *Decimals on Grids and Number Lines* Unit 8*: Growth Patterns* **Standard 2:** Unit 1*: Number Puzzles and Multiple Towers* Unit 7: *How Many People? How Many Teams* Unit 8: *Growth Patterns*
Everyday Math	**Standard 1:** *4.1, 4.2, 4.3, 4.8, 7.1, 7.2, 9.2* **Standard 2:** *7.3, 9.6, 9.7* **Standard 3:** *4.1,4.2,4.3,7.3,7.4,7.7,7.8,8.5,9.1,9.2,9.3,9.4,9.5,9.6,9.7,9.8,9.11,9.12,10.4,10.8* **Standard 4:** *4.1,4.2,.4.3,4.6,7.1,7.2,7.3,7.4,9.12*	**Standard 1:** Unit 3*: Multiplication and Divisions; Number Sentences and Algebra* Unit 5: *Big Numbers, Estimation, and Computation* **Standard 2:** Unit 4*: Decimals and Their Use* Unit 5*: Big Numbers, Estimation, and Computation* Unit 8: *Perimeter and Area* **Standard 3:** Unit 3: *Multiplication and Divisions; Number Sentences and Algebra* Unit 5*: Big Numbers, Estimation, and Computation* Unit 6: *Division; Map Ref Frames; Measure of Angles* Unit 8: *Perimeter and Area* Unit 9: *Fractions, Decimals, and Percent* Unit 11*: 3-D shapes, Weight, Volume, and Capacity* Unit 12: *Rates*	**Standard 1:** Unit 7: *Exponents and Negative Numbers* **Standard 2:** Unit 1: *Number Theory* Unit 2: *Estimation and Computation* Unit 4: *Division* Unit 7: *Exponents and Negative Numbers* Unit 10: *Algebraic Expressions*

Domain	**Operations and Algebraic Thinking Grade 3**	**Operations and Algebraic Thinking Grade 4**	**Operations and Algebraic Thinking Grade 5**
Cluster	***Understand properties of multiplication and the relationship between multiplication and division.***	***Gain familiarity with factors and multiples.***	
Standards	**3.OAT.5.** Apply properties of operations as strategies to multiply and divide. Examples: If 6 × 4 = 24 is known, then 4 × 6 = 24 is also known. (Commutative property of multiplication.) 3 × 5 × 2 can be found by 3 × 5 = 15, then 15 × 2 = 30, or by 5 × 2 = 10, then 3 × 10 = 30. (Associative property of multiplication.) Knowing that 8 × 5 = 40 and 8 × 2 = 16, one can find 8 × 7 as 8 × (5 + 2) = (8 × 5) + (8 × 2) = 40 + 16 = 56. (Distributive property.) **3.OAT.6.** Understand division as an unknown-factor problem. For example, find 32 ÷ 8 by finding the number that makes 32 when multiplied by 8.	**4.OAT.4.** Find all factor pairs for a whole number in the range 1-100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number is the range 1-100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1-100 is prime or composite.	
Content Elaborations	5. Students should build understanding of these properties through the construction and use of models. 6. Make sense of and emphasize the connection between division and multiplication.		
Expectations for Learning (Benchmark Indicators)			

Program Correlations			
Investigations	**Standard 5:** Unit 5: *Equal Groups* Unit 6: *Stories, Tables, and Graphs* Unit 7: *Finding Fair Shares* **Standard 6:** Unit 5: *Equal Groups*	**Standard 4:** Unit 1: *Factors, Multiples, and Arrays* Unit 3: *Multiple Towers and Division Stories*	
Everyday Math	**Standard 5:** *4.1, 4.2, 4.5, 4.6, 4.7, 7.2, 7.3, 9.2, 9.4, 9.6, 9.11, 9.12* **Standard 6:** *4.3, 4.4, 4.6, 7.3, 7.6, 9.1*	**Standard 4:** Unit 3: *Multiplication and Division; Number Sentences and Algebra* Unit 6: *Division; Map Ref Frames; Measure of Angles*	

Domain	Operations and Algebraic Thinking		
Cluster	***Multiply and divide within 100.***		
Standards	**3.OAT.7.** Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.		
Content Elaborations			
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Investigations	**Standard 7:** Unit 5: *Equal Groups* Unit 6: *Stories, Tables, and Graphs* Unit 7: *Finding Fair Shares* Unit 8: *How Many Hundreds? How Many Miles?*		
Everyday Math	**Standard 7:** *4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7, 4.8, 5.12, 7.2, 7.3, 7.4, 7.6, 9.6, 9.9*		

Domain	**Operations and Algebraic Thinking Grade 3**	**Operations and Algebraic Thinking Grade 4**	**Operations and Algebraic Thinking Grade 5**
Cluster	***Solve problems involving the four operations, and identify and explain patterns in arithmetic.***	***Generate and analyze patterns.***	***Analyze patterns and relationships.***
Standards	**3.OAT.8.** Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding. **3.OAT.9.** Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends	**4.OAT.5.** Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. For example, given the rule “Add 3” and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.	**5.OAT.3.** Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. For example, given the rule “Add 3” and the starting number 0, and given the rule “Add 6” and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.

Content Elaborations

8.This emphasis is on multi-step problems.

9.This speaks to students' ability to generalize about operations and applying what they know to new situations--for example, if children understand that even numbers can always be split into two groups or into groups of two, if that even number is added repeatedly to itself, the total will always be an even number because one is simply adding a number that is already made of groups of two, or that can evenly add to the existing groups of two present in the original total.

3. For example, work with this is idea might include a problem like the following: (modified from: http://www.ncpublicschools.org/docs/acre/standards/common-core-tools/unpacking/math/5th.pdf)
Both Sam and Terri have no fish. They both go fishing each day. Sam catches 2 fish each day. Terri catches 4 fish each day. How many fish do they have after each of the five days? Make a graph to represent the problem using the information in the table below.

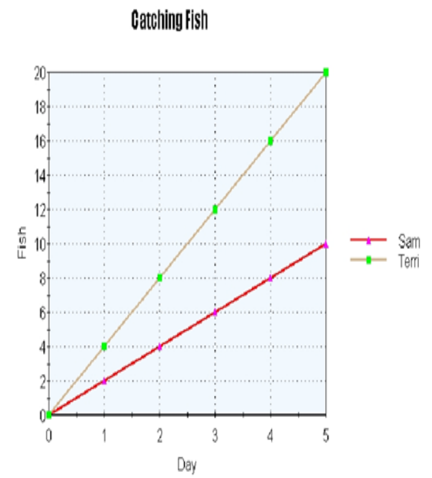
Days	Sam's Total Number of Fish	Terri's Total Number of Fish
0	0	0
1	2	4
2	4	8
3	6	12
4	8	16

If asked to describe the pattern, students may say something like:

Since Terri catches 4 fish each day, and Sam catches 2 fish, the amount of Terri's fish is always greater. Terri's fish is also always twice as much as Sam's fish.

A graph and discussion of same may look like the following:

"My graph shows that Terri always has more fish than Sam. Terri's fish increases

			at a higher rate since she catches 4 fish every day. Sam only catches 2 fish every day, so his number of fish increases at a smaller rate than Terri." *Important to note as well that the lines become increasingly further apart. Identify apparent relationships between corresponding terms. Additional relationships: The two lines will never intersect; there will not be a day in which boys have the same total of fish, explain the relationship between the number of days that has passed and the number of fish a boy has (2n or 4n, n being the number of days).*  In a related exercise, students might use the rule "start at zero and add 3" to write a sequence of numbers and write *"0, 3, 6, 9, 12, . . . "* and use the rule "start at zero and add 6" to write a sequence of numbers and write *"0, 6, 12, 18, 24, . . ."* After comparing these two sequences, the

			students notice that each term in the second sequence is twice the corresponding terms of the first sequence. One way they justify this is by describing the patterns of the terms. Their justification may include some mathematical notation (See example below). A student may explain that both sequences start with zero and to generate each term of the second sequence he/she added 6, which is twice as much as was added to produce the terms in the first sequence. Students may also use the distributive property to describe the relationship between the two numerical patterns by reasoning that 6 + 6 + 6 = 2 (3 + 3 + 3).
Expectations for Learning (Benchmark Indicators)			

Program Correlations			
Investigations	**Standard 8:** Unit 1: *Trading Stickers, Combining Coins* Unit 3: *Collections and Travel Stories* Unit 5: *Equal Groups* Unit 6: *Stories, Tables, and Graphs* Unit 8: *How Many Hundreds? How Many Miles?* Unit 9: Solids and Boxes **Standard 9:** Unit 1: *Trading Stickers, Combining Coins* Unit 3: *Collections and Travel Stories* Unit 5: *Equal Groups* Unit 6: *Stories, Tables, and Graphs* Unit 8: *How Many Hundreds? How Many Miles?*	**Standard 5:** Unit 8: *How Many Packages? How Many Groups?* Unit 9: *Penny Jars and Plant Growth*	**Standard 3:** Unit 8: *Growth Patterns*
Everyday Math	**Standard 8:** *2.7, 2.8, 2.9, 4.1, 7.4, 7.5, 7.7, 9.1, 9.5* **Standard 9:** *1.9. 2.1, 2.2, 4.5, 4.6, 7.1, 7.2*	**Standard 5:** Unit 3: *Multiplication and Division; Number Sentences and Algebra* Unit 10: *Reflections and Symmetry*	**Standard 3:** Unit 10: *Using Data and Algebraic Skills*

Domain	**Expressions and Equations Grade 6**	**Expressions and Equations Grade 7**	**Expressions and Equations Grade 8**
Cluster	***Apply and extend previous understandings of arithmetic to algebraic expressions.***	***Use properties of operations to generate equivalent expressions.***	***Work with radicals and integer exponents.***
Standards	**6.EE.1.** Write and evaluate numerical expressions involving whole-number exponents. **6.EE.2.** Write, read, and evaluate expressions in which letters stand for numbers. a. Write expressions that record operations with numbers and with letters standing for numbers. For example, express the calculation "Subtract y from 5" as 5 – y. b. Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. For example, describe the expression 2 (8 + 7) as a product of two factors; view (8 + 7) as both a single entity and a sum of two terms. c. Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole- number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). For example, use the formulas V = s^3 and A = 6 s^2 to find the volume and surface area of a cube with sides of length s = 1/2.	**7.EE.1.** Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients. **7.EE.2.** Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. For example, a + 0.05a = 1.05a means that "increase by 5%" is the same as "multiply by 1.05."	**8.EE.1.** Know and apply the properties of integer exponents to generate equivalent numerical expressions. For example, $3^2\times3^{-5}=3^{-3}=1/3^3=1/27$. **8.EE.2**. Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational. **8.EE.3.** Use numbers expressed in the form of a single digit times an integer power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other. For example, estimate the population of the United States as 3 × 10^8 and the population of the world as 7 × 10^9, and determine that the world population is more than 20 times larger. **8.EE.4.** Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology.

	6.EE.3. Apply the properties of operations to generate equivalent expressions. For example, apply the distributive property to the expression 3 (2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6 (4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y. **6.EE.4.** Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). For example, the expressions y + y + y and 3y are equivalent because they name the same number regardless of which number y stands for.		
Content Elaborations	The use of a dot (•) or parentheses between number terms is preferred. Less confusion will occur as students write algebraic expressions and equations if *x* represents only variables and not multiplication.	Have students build on their understanding of order of operations and use the properties of operations to rewrite equivalent numerical expressions that were developed in Grade 6. Students continue to use properties that were initially used with whole numbers and now develop the understanding that properties hold for integers, rational and real numbers. Student can interpret and explain in both English and mathematical terms the connections between the problem and the math symbolism, as well as the difference between two problems that seem similar on the surface. (See example on "Illustrative".)	Students now work with: positive and negative exponents, square and cube roots and scientific notation. It is no accident that these expectations are simultaneous, because it is the properties of counting-number exponents that provide the rationale for the properties of integer exponents. In other words, students should not be told these properties but rather should derive them through experience and reason.
Expectations for Learning (Benchmark Indicators)			

Program Correlations			
Connected Math	**Standard 1:** Investigation 4: *Prime Time* **Standard 2a:** Investigation 2-4: *Bits and Pieces II* Investigation 1-3: *Bits and Pieces III* CC Investigation 2: *Number Properties and Algebraic Equations - Bits & Pieces III* **Standard 2b:** Investigation 1, 3-5: *Prime Time* Investigation 2-4: *Bits and Pieces II* Investigation 1-3: *Bits and Pieces III* CC Investigation 2: *Number Properties and Algebraic Equations - Bits & Pieces III* **Standard 2c:** Investigation 1-5: *Covering and Surroundings* CC Investigation 2: *Number Properties and Algebraic Equations - Bits & Pieces III* **Standard 3:** CC Investigation 2: *Number Properties and Algebraic Equations - Bits & Pieces III* **Standard 4:** CC Investigation 2: *Number Properties and Algebraic Equations - Bits & Pieces III*	**Standard 1:** Investigation 3, 4: *Moving Straight Ahead* CC Investigation 2: *Equivalent Expressions - Moving Straight Ahead* **Standard 2:** CC Investigation 2: *Equivalent Expressions - Moving Straight Ahead*	**Standard 1:** Investigation 5: *Growing, Growing, Growing* **Standard 2:** Investigation 2, 3, 4: *Looking for Pythagoras* CC Investigation 1: *Negative Exponents - Growing, Growing, Growing* **Standard 3:** Investigation 1 ACE #39-40: *Growing, Growing, Growing* Investigation 2 ACE #15-17 Investigation 4 ACE #8 Investigation 5 ACE #56-60 **Standard 4:** Investigation 5 ACE #56, 57, 60: *Growing, Growing, Growing*

Domain	Expressions and Equations Grade 6	Expressions and Equations Grade 7	Expressions and Equations Grade 8
Cluster	***Reason about and solve one-variable equations and inequalities.***	***Solve real-life and mathematical problems using numerical and algebraic expressions and equations***	***Understand the connections between proportional relationships, lines, and linear equations.***
Standards	**6.EE.5.** Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true. **6.EE.6.** Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set. **6.EE.7.** Solve real-world and mathematical problems by writing and solving equations of the form x + p = q and px = q for cases in which p, q and x are all nonnegative rational numbers. **6.EE.8.** Write an inequality of the form x > c or x < c to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form x > c or x < c have infinitely many solutions; represent solutions of such inequalities on number line diagrams.	**7.EE.3.** Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. For example: If a woman making $25 an hour gets a 10% raise, she will make an additional 1/10 of her salary an hour, or $2.50, for a new salary of $27.50. If you want to place a towel bar 9 3/4 inches long in the center of a door that is 27 1/2 inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation. **7.EE.4.** Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities. a. Solve word problems leading to equations of the form px + q = r and p(x + q) = r, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width? b. Solve word problems leading to inequalities	**8.EE.5.** Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed. **8.EE.6.** Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation y = mx for a line through the origin and the equation y = mx + b for a line intercepting the vertical axis at b.

		of the form px + q > r or px + q < r, where p, q, and r are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem. For example: As a salesperson, you are paid $50 per week plus $3 per sale. This week you want your pay to be at least $100. Write an inequality for the number of sales you need to make, and describe the solutions.	
Content Elaborations	Students write equations for and solve "real world" problems with a variable. The process of translating between mathematical phrases and symbolic notation will also assist students in the writing of equations/inequalities for a situation. This process should go both ways; Students should be able to write a mathematical phrase for an equation. Additionally, the writing of equations from a situation or story does not come naturally for many students. A strategy for assisting with this is to give students an equation and ask them to come up with the situation/story that the equation could be referencing.	To assist students' assessment of the reasonableness of answers, especially problem situations involving fractional or decimal numbers, use whole-number approximations for the computation and then compare to the actual computation. Connections between performing the inverse operation and undoing the operations are appropriate here. It is appropriate to expect students to show the steps in their work. Students should be able to explain their thinking using the correct terminology for the properties and operations. Provide multiple opportunities for students to work with multi-step problem situations that have multiple solutions and therefore can be represented by an inequality. Students need to be aware that values can satisfy an inequality but not be appropriate for the situation, therefore limiting the solutions for that particular problem.	This cluster focuses on extending the understanding of ratios and proportions. Unit rates have been explored in Grade 6 as the comparison of two different quantities with the second unit a unit of one, (unit rate). In seventh grade unit rates were expanded to complex fractions and percents through solving multistep problems such as: discounts, interest, taxes, tips, and percent of increase or decrease. Proportional relationships were applied in scale drawings, and students should have developed an informal understanding that the steepness of the graph is the slope or unit rate. Now unit rates are addressed formally in graphical representations, algebraic equations, and geometry through similar triangles. Students are graphing proportional and/or linear relationships and interpreting/comparing the difference between them.
Expectations for Learning (Benchmark Indicators)			

Program Correlations			
Connected Math	**Standard 5:** Investigation 2-4: *Bits and Pieces II* Investigation 1-3: *Bits and Pieces III* Investigation 2 ACE #30-33: *Shapes and Designs* **Standard 6:** Investigation 3, 4: *Shapes and Designs* Investigation 5: *Covering and Surrounding* CC Investigation 2: *Number Properties and Algebraic Equations - Bits & Pieces III* **Standard 7:** Investigation 3, 4: *Shapes and Designs* Investigation 5: *Covering and Surrounding* CC Investigation 2*: Number Properties and Algebraic Equations - Bits & Pieces III* **Standard 8:** CC Investigation 3: *Integers and the Coordinate Plane - Bits & Pieces III*	**Standard 3:** Investigation 2-4: *Variables and Patterns* Investigation 1-4: *Accentuate the Negative* Investigation 1-4: *Moving Straight Ahead* **Standard 4a:** Investigation 1-3: *Variables and Patterns* Investigation 1-4: *Moving Straight Ahead* **Standard 4b:** Investigation 2 ACE #44: *Moving Straight Ahead* CC Investigation: *Inequalities – Moving Straight Ahead*	**Standard 5:** Investigation 2: *Thinking with Mathematical Models* CC Investigation 2: *Functions - Growing, Growing, Growing* **Standard 6:** Investigation 2: *Thinking with Mathematical Models* CC Investigation 2: *Functions - Growing, Growing, Growing*

Domain	**Expressions and Equations Grade 6**	**Expressions and Equations Grade 7**	**Expressions and Equations Grade 8**
Cluster	***Represent and analyze quantitative relationships between dependent and independent variables.***		***Analyze and solve linear equations and pairs of simultaneous linear equations.***
Standards	**6.EE.9**. Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation d = 65t to represent the relationship between distance and time.		**8.EE.7**. Solve linear equations in one variable. a. Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form x = a, a = a, or a = b results (where a and b are different numbers). b. Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms. Analyze and solve pairs of simultaneous linear equations. **8.EE.8.** Analyze and solve pairs of simultaneous linear equations. a. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously. b. Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. For example, 3x + 2y = 5 and 3x + 2y = 6 have no solution because 3x + 2y cannot simultaneously be 5 and 6. c. Solve real-world and mathematical problems leading to two linear equations in two variables. For example, given coordinates for two pairs of points, determine whether the line through the

			first pair of points intersects the line through the second pair.
Content Elaborations	The goal is to help students connect the pieces together. This can be done by having students use multiple representations for the mathematical relationship. Students need to be able to translate freely among the story, words (mathematical phrases), models, tables, graphs and equations. They also need to be able to start with any of the representations and develop the others. Students will: know the difference between dependent and independent variables, create a table and a graph to represent the relationship between variables and interpret results.		This cluster builds on the informal understanding of slope from graphing unit rates in Grade 6 and graphing proportional relationships in Grade 7 with a stronger, more formal understanding of slope. It extends solving equations to understanding solving systems of equations, or a set of two or more linear equations that contain one or both of the same two variables. Once again the focus is on a solution to the system. Most student experiences should be with numerical and graphical representations of solutions. Beginning work should involve systems of equations with solutions that are ordered pairs of integers, making it easier to locate the point of intersection, simplify the computation and hone in on finding a solution. More complex systems can be investigated and solve by using graphing technology. Contextual situations relevant to eighth graders will add meaning to the solution to a system of equations. Students should explore many problems for which they must write and graph pairs of equations leading to the generalization that finding one point of intersection is the single solution to the system of equations. Provide opportunities for students to connect the solutions to an equation of a line, or solution to a system of equations, by graphing, using a table and writing an equation. Students should receive opportunities to compare equations and systems of equations, investigate using graphing calculators or graphing utilities, explain differences verbally and in writing, and use models such as equation balances. Problems should be structured so that students also experience equations that represent parallel lines and equations that are equivalent. This will help them to begin to understand the

			relationships between different pairs of equations: When the slope of the two lines is the same, the equations are either different equations representing the same line (thus resulting in many solutions), or the equations are different equations representing two not intersecting, parallel, lines that do not have common solutions. System-solving in Grade 8 should include estimating solutions graphically, solving using substitution, and solving using elimination. Students again should gain experience by developing conceptual skills using models that develop into abstract skills of formal solving of equations. Provide opportunities for students to change forms of equations (from a given form to slope-intercept form) in order to compare equations.
Expectations for Learning (Benchmark Indicators)			
Program Correlations			
Connected Math	**Standard 9:** Investigation 2: *Data About Us* Investigation 2: *Covering and Surrounding* CC Investigation 2: *Number Properties - Bits & Pieces III*		**Standard 7a:** CC Investigation 2: *Functions - Growing, Growing, Growing* **Standard 7b:** Investigation 2: *Thinking with Mathematical Models* Investigation 1-4: *Say it with Symbols* **Standard 7c:** Investigation 2: *Thinking with Mathematical Models* Investigation 1-3: *Say it with Symbols* **Standard 8a:** Investigation 2-4: *Shapes of Algebra* **Standard 8b:** Investigation 1-4: *Shapes of Algebra*

			Standard 8c: Investigation 2-4: *Shapes of Algebra*

Domain	**Counting and Cardinality** **Kindergarten**
Cluster	***Know number names and the count sequence***
Standards	**K.CC.1.** Count to 100 by ones and by tens. **K.CC.2.** Count forward beginning from a given number within the known sequence (instead of having to begin at 1). **K.CC.3.** Write numbers from 0 to 20. Represent a number of objects with a written numeral 0-20 (with 0 representing a count of no objects).
Content Elaborations	1. Understanding of difference between -ty and -teen number words (auditory discrimination) 3. Forming numbers from left to right should be emphasized (I.e. 19 needs to be written as 1 then 9 from left to right.
Expectations for Learning (Benchmark Indicators)	
Program Correlations	
Investigations	**Standard 1:** All Units **Standard 2:** Unit 3: *What Comes Next?* Unit 5: *Make a Shape, Build a Block* Unit 6: *How Many Do You Have?* Unit 7: *Sorting and Surveys* **Standard 3:** Unit 1: *Who Is in School Today?* Unit 2: *Counting and Comparing* Unit 4: *Measuring and Counting* Unit 6: *How Many Do You Have?*

Everyday Math	
Connections	

Domain	**Counting and Cardinality** **Kindergarten**
Cluster	***Count to tell the number of objects***
Standards	**K.CC.4.** Understand the relationship between numbers and quantities; connect counting to cardinality. a. When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object. b. Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted. c. Understand that each successive number name refers to a quantity that is one larger. **K.CC.5**. Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.
Content Elaborations	4. Cardinality of a number means the quantity (i.e. the number four means the quantity of 4; the number written or spoken represents the size of the set.
Expectations for Learning (Benchmark Indicators)	
Program Correlations	
Investigations	**Standard 4:** All Units **Standard 5:** All Units
Everyday Math	
Connections	

Domain	**Counting and Cardinality** **Kindergarten**
Cluster	**Compare Numbers**
Standards	**K.CC.6.** Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies. **K.CC.7.** Compare two numbers between 1 and 10 presented as written numerals.
Content Elaborations	6. The use of greater than/less than should be used rather than bigger or smaller. 7. Using manipulatives; pictorial representation and recording sheet [I have blank counters. I have (more than/less than/ the same as) my partner. My partner has blank counters.]
Expectations for Learning (Benchmark Indicators)	
Program Correlations	
Investigations	**Standard 6:** Unit 2: *Counting and Comparing* Unit 3: *What Comes Next?* Unit 4: *Measuring and Counting* Unit 5: *Make a Shape, Build a Block* Unit 6: *How Many Do You Have?* Unit 7: *Sorting and Surveys* **Standard 7**: Unit 2: *Counting and Comparing* Unit 4: *Measuring and Counting* Unit 6: *How Many Do You Have?*
Everyday Math	
Connections	

Domain	**Number and Operations - Fractions Grade 3**	**Number and Operations - Fractions Grade 4**	**Number and Operations - Fractions Grade 5**
Cluster	***Develop understanding of fractions as numbers.***	***Extend understanding of fractions equivalence and ordering.***	***Use equivalent fractions as a strategy to add and subtract fractions.***
Standards	**3.NOF.1.** Understand a fraction 1/b as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size 1/b. **3.NOF.2.** Understand a fraction as a number on the number line; represent fractions on a number line diagram. a. Represent a fraction 1/b on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size 1/b and that the endpoint of the part based at 0 locates the number 1/b on the number line. b. Represent a fraction a/b on a number line diagram by marking off a lengths 1/b from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line. **3.NOF.3.** Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size. a. Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line. b. Recognize and generate simple equivalent fractions, e.g., 1/2 = 2/4, 4/6 = 2/3). Explain why the fractions are equivalent, e.g., by using a visual fraction model. c. Express whole numbers as fractions,	**4.NOF.1.** Explain why a fraction a/b is equivalent to a fraction (n × a)/(n × b) by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions. **4.NOF.2.** Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.	**5.NOF.1.** Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. For example, 2/3 + 5/4 = 8/12 + 15/12 = 23/12. (In general, a/b + c/d = (ad + bc)/bd.) **5.NOF.2.** Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. For example, recognize an incorrect result 2/5 + 1/2 = 3/7, by observing that 3/7 < 1/2.

	and recognize fractions that are equivalent to whole numbers. Examples: Express 3 in the form 3 = 3/1; recognize that 6/1 = 6; locate 4/4 and 1 at the same point of a number line diagram. d. Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.		
Content Elaborations	1 through 3 Please note, specify the meaning of the whole. The emphasis is on conceptual understanding of unit fractions through the use of models. In second grade the area model (2.G.3) was introduced and should be built upon and linked to the linear model (number line). All operations with fractions that come in 4th and 5th grade are based on unit fraction understanding.	1. Equivalent fractions are generated through the use of a variety of visual fraction models (area and linear) 1. Remember that n/n is equal to 1 whole and that equivalent fractions are found by multiplying by different versions of 1 whole (6/6 or 12/12 for example).	1. It is not necessary to find the least common denominator to calculate sums and differences and in fact the effort of finding the least common denominator distracts from understanding algorithms for adding and subtracting fractions. Note, the commonness of the denominator is what matters.
Expectations for Learning (Benchmark Indicators)			

Program Correlations			
Investigations	**Standard 1**: Unit 7: *Finding Fair Shares* **Standard 2:** Unit 7: *Finding Fair Shares* **Standard 3:** Unit 7: *Finding Fair Shares*	**Standard 1:** Unit 6: *Fraction Cards and Decimal Squares* **Standard 2:** Unit 6: *Fraction Cards and Decimal Squares*	**Standard 1:** Unit 4: *What's the Portion* **Standard 2:** Unit 4: *What's the Portion* Unit 7: *How Many People?*
Everyday Math	**Standard 1:** Unit 5, Unit 8, Unit 11 **Standard 2:** Unit 8 **Standard 3:** Unit 8, Unit 9	**Standard 1:** Unit 7: *Fractions and Their Use; Probability* **Standard 2:** Unit 7: *Fractions and Their Use; Probability*	**Standard 1:** Unit 5: *Fractions, Decimals, Percents* Unit 6*: Using Data and Add and Sub Fractions* Unit 8: *Fractions and Ratios* **Standard 2:** Unit 5: *Fractions, Decimals, Percents* Unit 6: *Using Data and Add and Sub Fractions* Unit 8: *Fractions and Ratios*

Domain	**Number and Operations - Fractions Grade 3**	**Number and Operations - Fractions Grade 4**	**Number and Operations - Fractions Grade 5**
Cluster		***Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.***	***Apply and extend previous understandings of multiplication and division to multiply and divide fractions.***
Standards		**4.NOF.3.** Understand a fraction a/b with a > 1 as a sum of fractions 1/b. a. Understand addition and subtraction of fractions as joining and separating parts referring to the same whole. b. Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model. Examples: 3/8 = 1/8 + 1/8 + 1/8 ; 3/8 = 1/8 + 2/8 ; 2 1/8 = 1 + 1 + 1/8 = 8/8 + 8/8 + 1/8. c. Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction. d. Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem. **4.NOF.4.** Apply and extend previous understandings of multiplication to multiply a fraction by a whole number. a. Understand a fraction a/b as a multiple of 1/b. For example, use a visual fraction model to represent 5/4 as the product 5 × (1/4), recording the conclusion by the equation 5/4 = 5 × (1/4). b. Understand a multiple of a/b as a multiple of 1/b, and use this understanding to multiply	**5.NOF.3**. Interpret a fraction as division of the numerator by the denominator (a/b = a ÷ b). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. For example, interpret 3/4 as the result of dividing 3 by 4, noting that 3/4 multiplied by 4 equals 3, and that when 3 wholes are shared equally among 4 people each person has a share of size 3/4. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie? **5.NOF.4** Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction. a. Interpret the product (a/b) × q as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations a × q ÷ b. For example, use a visual fraction model to show (2/3) × 4 = 8/3, and create a story context for this equation. Do the same with (2/3) × (4/5) = 8/15. (In general, (a/b) × (c/d) = ac/bd.) b. Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular

		a fraction by a whole number. For example, use a visual fraction model to express 3 × (2/5) as 6 × (1/5), recognizing this product as 6/5. (In general, n × (a/b) = (n × a)/b.) c. Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem. For example, if each person at a party will eat 3/8 of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie?	areas. **5.NOF.5.** Interpret multiplication as scaling (resizing), by: a. Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication. b. Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence a/b = (n×a)/(n×b) to the effect of multiplying a/b by 1. **5.NOF.6.** Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem. **5.NOF.7.** Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions. a. Interpret division of a unit fraction by a non-zero whole number, and compute such quotients. For example, create a story context for (1/3) ÷ 4, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that (1/3) ÷ 4 = 1/12 because (1/12) × 4 = 1/3. b. Interpret division of a whole number by a unit fraction, and compute such quotients. For example, create a story context for 4 ÷ (1/5), and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that 4 ÷ (1/5) = 20 because 20 × (1/5) = 4. c. Solve real world problems involving division

			of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem. For example, how much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 1/3-cup servings are in 2 cups of raisins?
Content Elaborations		3a. This is referring to the sum of 1/b; 'a' number of times. 3. Link the decomposition of whole numbers to the decomposition of fractions (including mixed numbers) with a numerator greater than one and justify with a fractional model. 3c. Students need to build an understanding from "two and one-third" to 2 + ⅓ to 2 ⅓ . 3c. Properties of operations refers to both the commutative property as well as the nature of the operations (ie; given the problem ⅞ - ⅜, they might interpret it as a removal situation but make use of what they know about the relationship between addition and subtraction to add up from ⅜ to find that difference; or they might interpret it as a comparative situation and add up or remove to find the difference).	3. Particular consideration needs to be paid to develop student understanding of ¾ as 3 divided by 4. Straight up division should be represented in all ways: ¾ or 3÷4 or $4\overline{)3}$ 4a. A common misconception when multiplying a fraction by a whole number is to multiply both the numerator and denominator by the whole number. For example: ⅖ x 3= (2x3) / (5x3) = 6/15 5. In preparation for grade 6 work in ratios and proportional reasoning, students start to see products such as 5 x 3 or ½ x 3 as expressions that can be interpreted in terms of a quantity 3 and a scaling factor of 5 or ½. In addition to knowing that 5 x 3=15 they can also say that 5 x 3 is 5 times as big as 3 and ½ x 3 as half the size of 3. 7. Present problem situations and have students use multiple models and equations to solve the problem. It is important for students to develop conceptual understanding of multiplication and division of fractions through contextual situations. 7b. 4 ÷ ⅕ … How many groups of ⅕ can be found in 4?
Expectations for Learning (Benchmark			

Indicators)			
Program Correlations			
Investigations		**Standard 3:** Unit 6: *Fraction Cards & Decimal Squares* **Standard 4:** Unit 6: *Fraction Cards & Decimal Squares*	**Standard 3:** Unit 6: *Decimals on Grids and Number Lines* **Standard 4:** Unit 4: *What's that Portion* **Standard 5:** Unit 4*: What's that Portion* **Standard 6:** Unit 4: *What's that Portion* Unit 9: *How Long Can You Stand* **Standard 7:** Unit 4: *What's that Portion*
Everyday Math		**Standard 3**: Unit 7: *Fractions and Their Use; Probability* **Standard 4**: Unit 7: *Fractions and Their Use; Probability*	**Standard 3:** Unit 5: *Fractions, Decimals, Percents* Unit 6: *Using Data and Add and Sub Fractions* **Standard 4:** Unit 8: *Fractions and Ratios* Unit 9: *Coor, Area, Perimeter, Volume* **Standard 5:** Unit 1: *Number Theory* Unit 4: *Division* Unit 6: *Using Data and Add and Sub Fractions* Unit 8: *Fractions and Ratios* **Standard 6:** Unit 8: *Fractions and Ratios* **Standard 7:** Unit 8: *Fractions and Ratios* Unit 9: *Coor, Area, Perimeter, Volume* Unit 12: *Prob. Ratios and Rates*

Domain	**Number and Operations - Fractions Grade 3**	**Number and Operations - Fractions Grade 4**	**Number and Operations - Fractions Grade 5**
Cluster		***Understand decimal notations for fractions, and compare decimal fractions.***	
Standards		**4.NOF.5**. Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100. For example, express 3/10 as 30/100, and add 3/10 + 4/100 = 34/100. **4.NOF.6.** Use decimal notation for fractions with denominators 10 or 100. For example, rewrite 0.62 as 62/100; describe a length as 0.62 meters; locate 0.62 on a number line diagram. **4.NOF.7.** Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual model.	
Content Elaborations		Standards 5, 6 &7 It is important to scaffold understanding of this concept through the use of models and manipulatives (for example a decimal square or meter stick) to show equivalence between tenths and hundredths.	
Expectations for Learning (Benchmark Indicators)			

Program Correlations			
Investigations		**Standard 5:** Unit 6: *Fraction Cards & Decimal Squares* **Standard 6:** Unit 6: *Fraction Cards & Decimal Squares* **Standard 7:** Unit 6: *Fraction Cards & Decimal Squares* Unit 7: *Moving Between Solids and Silhouettes*	
Everyday Math		**Standard 5:** Unit 7: *Fractions and Their Use; Probability* Unit 9: *Fractions, Decimals, and Percent* Unit 10: *Reflections and Symmetry* **Standard 6:** Unit 4: *Decimals and Their Use* Unit 9: *Fractions, Decimals, and Percents* **Standard 7:** Unit 4: *Big Numbers, Estimation, and Computation*	

Domain	**Ratio and Proportional Relationships Grade 6**	**Ratio and Proportional Relationships Grade 7**
Cluster	***Understand ratio concepts and use ratio reasoning to solve problems***	***Analyze proportional relationships and use them to solve real-world and mathematical problems.***
Standards	**6.RPR.1.** Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes." **6.RPR.2.** Understand the concept of a unit rate a/b associated with a ratio a:b with b ≠ 0, and use rate language in the context of a ratio relationship. For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is 3/4 cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger." **6.RPR.3.** Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations. a. Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios. b. Solve unit rate problems including those involving unit pricing and constant speed. For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed? c. Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent. d. Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying and dividing quantities.	**7.RPR.1.** Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. For example, if a person walks 1/2 mile in each 1/4 hour, compute the unit rate as the complex fraction 1/2/1/4 miles per hour, equivalently 2 miles per hour. **7.RPR.2.** Recognize and represent proportional relationships between quantities. a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin. b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships. c. Represent proportional relationships by equations. For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship between the total cost and the number of items can be expressed as t = pn. d. Explain what a point (x, y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0, 0) and (1, r) where r is the unit rate. **7.RPR.3.** Use proportional relationships to solve multi-step ratio and percent problems. Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error.
Content Elaborations	1. Fractions and ratios may represent different comparisons. Fractions always express a part-to-whole comparison, but ratios can express a part-to-whole comparison or a part-to-part comparison. For example: ⅔ of the class are girls, this means that the ratio of girls to boys in the class is 2:1.	1. Fractions and ratios may represent different comparisons. Fractions always express a part-to-whole comparison, but ratios can express a part-to-whole comparison or a part-to-part comparison. For example: ⅔ of the class are girls, this means that the ratio of girls to boys in the class is 2:1.

	2. Please note - for any given ratio there are two unit rates (for example: miles per hour and hours per mile) 3b. Although algorithms provide efficient means for finding solutions, the cross-product algorithm commonly used for solving proportions will not aid in the development of proportional reasoning. Delaying the introduction of rules and algorithms will encourage thinking about multiplicative situations instead of indiscriminately applying rules. Multiplicative reasoning is used when finding the missing element in a proportion. 3c. Percents are often taught in relationship to learning fractions and decimals. This cluster indicates that percents are to be taught as a special type of rate. Provide students with opportunities to find percents in the same ways they would solve rates and proportions. 3c. Often there is a misunderstanding that a percent is always a natural number less than or equal to 100. Provide examples of percent amounts that are greater than 100%, and percent amounts that are less 1%.	1-3. Providing opportunities to solve problems based within contexts that are relevant to seventh graders will connect meaning to rates, ratios and proportions. 1 and 2. Unit-rate problems move from whole number comparisons to more sophisticated numbers: fractions per fractions. 2b. Proportional relationships are further developed through the analysis of graphs, tables, equations and diagrams. This is not the time for students to learn to cross multiply to solve problems. 3. Percents have been introduced as rates in Grade 6 and should continue to follow the thinking involved with rates and proportions. Solutions to problems can be found by using the same strategies for solving rates, such as looking for equivalent ratios or based upon understandings of decimals. Previously, percents have focused on "out of 100"; now percents above 100 are encountered.
Expectations for Learning (Benchmark Indicators)		
Program Correlations		
Connected Math	**Standard 1:** Investigation 4: *Bits and Pieces I* **Standard 2:** CC Investigation 1: *Rates and Ratios - Bits & Pieces III* **Standard 3a. and 3b.:** Investigation 1: *Bits and Pieces I* Investigation 4 CC Investigation 1: *Rates and Ratios* **Standard 3c.:** Investigation 4, 5: *Bits and Pieces III* **Standard 3d.** CC Investigation 1: *Rates and Ratios*	**Standard 1:** CC Investigation 1: *Graphing Proportions* **Standard 2a:** Investigation 4: *Comparing and Scaling* CC Investigation 1: *Graphing Proportions* **Standard 2b.:** Investigation 3, 4: *Comparing and Scaling* Investigation 1-4: *Moving Straight Ahead* **Standard 2c.:** Investigation 1-3, 4 ACE #13: *Variable and Patterns* Investigation 4: *Comparing and Scaling* Investigation 4, 5: *Stretching and Shrinking* Investigation 1-4: *Moving Straight Ahead*

		Standard 2d.: Investigation 2: *Variable and Patterns* Investigation 3 ACE #9, #10: *Comparing and Scaling* Investigation 1-4: *Moving Straight Ahead* CC Investigation 1: *Graphing Proportions* **Standard 3:** Investigation 4 ACE #12: *Variable and Patterns* Investigation 1-4: *Comparing and Scaling* Investigation 4, 5: *Stretching and Shrinking*

Domain	**Functions** **Grade 8**
Cluster	**Define, evaluate, and compare functions.**
Standards	**8.F.1.** Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. **8.F.2.** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change. **8.F.3.** Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. For example, the function $A = s^2$ giving the area of a square as a function of its side length is not linear because its graph contains the points (1,1), (2,4) and (3,9), which are not on a straight line.
Content Elaborations	In grade 6, students plotted points in all four quadrants of the coordinate plane. They also represented and analyzed quantitative relationships between dependent and independent variables. In Grade 7, students decided whether two quantities are in a proportional relationship. In Grade 8, students begin to call relationships functions when each input is assigned to exactly one output. Also, in Grade 8, students learn that proportional relationships are part of a broader group of linear functions, and they are able to identify whether a relationship is linear. Nonlinear functions are included for comparison. To determine whether a relationship is a function, students should be expected to reason from a context, a graph, or a table, after first being clear which quantity is considered the input and which is the output. When a relationship is not a function, students should produce a counterexample: an "input value" with at least two "output values." If the relationship is a function, the students should explain how they verified that for each input there was exactly one output. The "vertical line test" should be avoided because (1) it is too easy to apply without thinking, (2) students do not need an efficient strategy at this point, and (3) it creates misconceptions for later mathematics, when it is useful to think of functions more broadly, such as whether x might be a function of y. The standards explicitly call for exploring functions numerically, graphically, verbally, and algebraically (symbolically, with letters). For fluency and flexibility in thinking, students need experiences translating among these. In Grade 8, the focus is on linear functions, and students begin to recognize a linear function from its form $y = mx + b$. Students also need experiences with nonlinear functions, including functions given by graphs, tables, or verbal descriptions but for which there is no formula for the rule, such as a girl's height as a function of her age.
Expectations for Learning (Benchmark Indicators)	

Program Correlations	
Connected Math	**Standard 1:** CC Investigation 2: *Functions - Growing, Growing, Growing* **Standard 2:** Investigation 1: *Thinking with Mathematical Models* Investigation 1 ACE #25,#26, #38, #47: *Growing, Growing, Growing* Investigation 2-4: *Frogs, Fleas, and Painted Cubes* Investigation 2: *Say it with Symbols* **Standard 3:** Investigation 2, 3, 5: *Thinking with Mathematical Models* Investigation 3: *Growing, Growing, Growing* Investigation 4: *Shapes of Algebra* Investigation 4: *Say it with Symbols*

Domain	Functions Grade 8
Cluster	***Use functions to model relationships between quantities.***
Standards	**8.F.4.** Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values. **8.F.5.** Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.
Content Elaborations	Students need to graph data and look for patterns, then generalize and symbolically represent the patterns (create a function). They also need opportunities to draw graphs (qualitatively, based upon experience) representing real-life situations with which they are familiar. Some students may still need clarification around misconceptions: • When input values are not increasing consecutive integers (e.g., when the input values are decreasing, when some integers are skipped, or when some input values are not integers), some students have more difficulty identifying the pattern and calculating the slope. • Some students may not pay attention to the scale on a graph, assuming that the scale units are always "one." • Some students graph incorrectly because they don't understand that x usually represents the independent variable and y represents the dependent variable. Emphasize that this is a convention that makes it easier to communicate.
Expectations for Learning (Benchmark Indicators)	
Program Coorelations	
Connected Math	**Standard 4:** Investigation 1-3: *Thinking with Mathematical Models* Investigation 4: *Shapes of Algebra* Investigation 4: *Say it with Symbols* **Standard 5:** Investigation 2: *Thinking with Mathematical Models* Investigation 1-4: *Growing, Growing, Growing* Investigation 1-4: *Frogs, Fleas, Painted Cubes* Investigation 4: *Say it with Symbols*

COURSE: Algebra I

Interpreting Categorical and Quantitative Data

Summarize, represent, and interpret data on a single count or measurement variable.

1. Represent data with plots on the real number line (dot plots, histograms, and box plots).

2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.

3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).

4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.

6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.

a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data.

Use given functions or choose a function suggested by the context. Emphasize linear and exponential models.

b. Informally assess the fit of a function by plotting and analyzing residents.

c. Fit a linear function for a scatter plot that suggests a linear association.

Interpret Linear Models

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.

8. Compute (using technology) and interpret the correlation coefficient of a linear fit.

9. Distinguish between correlation and causation.

Making Inferences and Justifying Conclusions

Understand and evaluate random processes underlying statistical experiments

1. Understand statistics as a process for making inferences about population parameters based on a random sample from that population.

2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. would a result of 5 tails in a row cause you to question the model?*

Make inferences and justify conclusions from sample surveys, experiments, and observational studies

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.

4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.

5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.

6. Evaluate reports based on data.

Conditional Probability and the Rules of Probability

Understand independence and conditional probability and use them to interpret data

1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or,""and," "not").

2. Understand that two events **A** and **B** are independent if the probability of **A** and **B** occurring together is the product of their probabilities, and use this characterization to determine if they are independent.

3. Understand the conditional probability of **A** given **B** as **P (A and B)/P (B),** and interpret independence of **A** and **B** as saying that the conditional probability **A** given **B** is the same as the probability of **A,** and the conditional probability of **B** given **A** is the same as the probability of **B.**

4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect: data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.*

5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.*

Use the rules of probability to compute probabilities of compound events in a uniform probability model

6. Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model.

7. Apply the Addition Rule, P(A or B) = P(A) + P(B) – P A and B), and interpret the answer in terms of the model.

8. (+) Apply the general Multiplication Rule in a uniform probability model, P(A and B) = P(A)P(B|A) = P(B)P(A|B), and interpret the answer in terms of the model.

9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

Using Probability to Make Decisions

Calculate expected values and use them to solve problems

1. (+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.

2. (+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution.

3. (+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. *For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choice, and find the expected grade under various grading schemes.*

4. (+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. *For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?*

Use probability to evaluate outcomes of decisions

5. (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

a. Find the expected payoff for a game of chance. *For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.*

b. Evaluate and compare strategies on the basis of expected values. *For example, compare a* *high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.*

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

7. (+) Analyze decisions and strategies, using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game.)

Expressing Geometric Properties with Equations

Use coordinates to prove simple geometric theorems algebraically

4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point* $(1, \sqrt{3})$ *lies on the circle centered at the origin and containing the point (0, 2).*

5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

Real Numbers

Cluster Reason quantitatively and use units to solve problems

1. Standards Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

2. Define appropriate quantities for the purpose of descriptive modeling. Choose a level of accuracy appropriate to limitations on measurement reporting quantities

3. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

Seeing Structure in Expressions

Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.

(a) Interpret parts of an expression, such as terms, factors, and coefficients.

(b) Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret P(1 + r)n as the product of P and a factor not depending on P.

2. Use the structure of an expression to identify ways to rewrite it. For example,

see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.

Write expressions in equivalent forms to solve problems

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression

a. Factor a quadratic expression to reveal the zeros of the function it defines

Arithmetic with Polynomials and Rational Expressions

Perform arithmetic operations on polynomials

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Understand the relationship between zeros and factors of polynomials

3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Creating Equations

Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable in a modeling context. For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.

4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. For example, rearrange Ohm’s law V = IR to highlight resistance R.

Reasoning with Equations and Inequalities

Understand solving equations as a process of reasoning and explain the reasoning

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

Solve systems of equations

5. Prove that a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.

Represent and solve equations and inequalities graphically

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where f(x) and/or g(x) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

12. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Interpreting Functions

Understand the concept of a function and use function notation

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If *f* is a function and *x* is an element of its domain, the *f(x)* denotes the output of *f* corresponding to the input *x.* The graph of *f* is the graph of the equation $y = f(x)$.

2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

a. Use the process of factoring

Building Functions

Build a function that models a relationship between two quantities

1. Write a function that describes a relationship between two quantities.

a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

Build new functions from existing functions

3. Identify the effect on the graph of replacing *f(x)* by *f(x) + k, k f(x), f(kx),* and *f(x + k)* for specific values of *k* (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Linear and Exponential Models

Construct and compare linear and exponential models and solve problems

1. Distinguish between situations that can be modeled with linear functions and with exponential functions.

a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.

b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

2a. Construct linear and exponential functions

3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.

Interpret expressions for functions in terms of the situation they model

5. Interpret the parameters in a linear or exponential function in terms of a context.

COURSE: Geometry

Congruence

Experiment with transformations in the plane

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Understand congruence in terms of rigid motions

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Prove geometric theorems

9. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*

10. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*

11. Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

Make geometric constructions

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*

13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Similarity, Right Triangles, and Trigonometry

Understand similarity in terms of similarity transformations

1. Verify experimentally the properties of dilations given by a center and a scale factor:

a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.

b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

3. Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

Prove theorems involving similarity

4. Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.*

5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Define trigonometric ratios and solve problems involving right triangles

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

7. Explain and use the relationship between the sine and cosine of complementary angles.

8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

Apply trigonometry to general triangles

9. (+) Derive the formula $A = 1/2\ ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.

11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

+ means for honors students

Circles

Understand and apply theorems about circles

1. Prove that all circles are similar.

2. Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*

3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

4. (+) Construct a tangent line from a point outside a given circle to the circle.

Find arc lengths and areas of sectors of circles

5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Expressing Geometric Properties with Equations

Translate between the geometric description and the equation for a conic section

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Use coordinates to prove simple geometric theorems algebraically

7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.

Geometric Measurement and Dimension

Explain volume formulas and use them to solve problems

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*

2. (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

Visualize relationships between two-dimensional and three-dimensional objects

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).

3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

Real Numbers

Cluster Reason quantitatively and use units to solve problems

1. Standards Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

2. Define appropriate quantities for the purpose of descriptive modeling. Choose a level of accuracy appropriate to limitations on measurement reporting quantities

3. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

Interpreting Functions

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Linear and Exponential Models

Construct and compare linear and exponential models and solve problems

2a. Construct linear and exponential functions, (2b) including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table.)

Trigonometric Functions

Extend the domain of trigonometric functions using the unit circle

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

3. (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$, and $\pi/6$, and use the unit circle to express the values of sine, cosines, and tangent for x, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.

4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Model periodic phenomena with trigonometric functions

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

Prove and apply trigonometric identities

8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to calculate trigonometric ratios.

COURSE: Algebra II

Number & Quantity High School

Real Numbers

Extend the properties of exponents to rational exponents

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.

2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Cluster Reason quantitatively and use units to solve problems

1. Standards Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

2. Define appropriate quantities for the purpose of descriptive modeling. Choose a level of accuracy appropriate to limitations on measurement reporting quantities.

3. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

The Complex Number System

Perform arithmetic operations with complex numbers

1. Know there is a complex number i such as i^2 = -1, and every complex number has the form a +bi with a and b real.
2. Use the relation i^2 = -1 and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Represent complex numbers and their operations on the complex plane

4. Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

5. Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. For example, (1 - √3i)3 = 8

because (1 - √3i) has modulus 2 and argument 120°.

6. Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

Use complex numbers in polynomial identities and equations

7. Solve quadratic equations with real coefficients that have complex solutions

Seeing the structure of expressions

Write expressions in equivalent forms to solve problems

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

c. Use the properties of exponents to transform expressions for exponential functions. For example the expression 1.15^t can be written as $(1.15^{\frac{1}{12}})^{12t} \approx 1.01212^t$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. For example, calculate mortgage payments.

Arithmetic with Polynomials and Rational Expressions

Understand the relationship between zeros and factors of polynomials

2. Know and apply the Remainder Theorem: For a polynomial p(x) and a number a, the remainder on division by x – a is p(a), so p(a) = 0 if and only if (x – a) is a factor of p(x).

3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems

4. Prove polynomial identities and use them to describe numerical relationships. For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.

Rewrite rational expressions

6. Rewrite simple rational expressions in different forms; write a(x)/b(x) in the form q(x) + r(x)/b(x), where a(x), b(x), q(x), and r(x) are polynomials with the degree of r(x) less than the degree of b(x), using inspection, long division, or, for the more complicated examples, a computer algebra system.

Reasoning with Equations and Inequalities

Understand solving equations as a process of reasoning and explain the reasoning

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Solve equations and inequalities in one variable

4. Solve quadratic equations in one variable.

a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

b. Solve quadratic equations by inspection (e.g., for x2 = 49), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them in a ± bi for real numbers a and b.

Represent and solve equations and inequalities graphically

11. Explain why the x-coordinates of the points where the graphs of the equations y = f(x) and y = g(x) intersect are the solutions of the equation f(x) = g(x); find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where f(x) and/or g(x) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

Interpreting Functions

Understand the concept of a function and use function notation

3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by F(0) = F(1) = 1, f(n + 1) = f(n) + f (n – 1) for n* ≥ 1.

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

b. Use the properties of exponents to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as $y = (1.02)^5$, $y = (0.95)^5$, $y = (1.01)^{12t}$, $y = (1.2)^{\frac{t}{10}}$, and classify them as representing exponential growth or decay.

Building Functions

Build a function that models a relationship between two quantities

1. Write a function that describes a relationship between two quantities.

(b) Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential and relate these functions to the model.*

Build new functions from existing functions

4. Find inverse functions.

a. Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *for* $x>0$ *or* $f(x) = (x+1)/(x-1)$ *for* $x \neq 1$.

Linear and Exponential Models

Construct and compare linear and exponential models and solve problems

2b. Including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table.)

4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where *a, c,* and *d* are numbers and the base *b* is 2, 10, or *e;* evaluate the logarithm using technology.

Interpret expressions for functions in terms of the situation they model

5. Interpret the parameters in a linear or exponential function in terms of a context.

Congruence

Experiment with transformations in the plane

5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

COURSE: Algebra II - Honors

The Complex Number System

Perform arithmetic operations with complex numbers

1. Know there is a complex number i such as i^2 = -1, and every complex number has the form a +bi with a and b real.

2. Use the relation i^2 = -1 and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

3. (+) Find the conjugate of complex number; use conjugates to find moduli and quotients of complex numbers. (This standard is suggested to be considered part of the next cluster).

Represent complex numbers and their operations on the complex plane

4. Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

5. Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. For example, (1 - √3i)3 = 8 because (1 - √3i) has modulus 2 and argument 120°.

6. Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

Use complex numbers in polynomial identities and equations

7. Solve quadratic equations with real coefficients that have complex solutions

8. (+) Extend polynomial identities to the complex numbers. For example, rewrite x^2 + 4 as (x +2i)(x - 2i)

9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Vector and Matrix Quantities

Represent and model with vector quantities

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., v, IvI, IIvII, v).

2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

Perform operations on vectors

3. (+) Solve problems involving velocity and other quantities that can be represented by vectors (Moved from N-VM cluster 1)

4. (+) Add and subtract vectors.

a. Add vectors end – to – end, component wise and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.

b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.

c. Understand vector subtraction v – w as v + (-w) is the additive inverse of w, with the same magnitude as w and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component - wise.

5. (+) Multiply a vector by a scalar

a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their directions; perform scalar multiplication component – wise, e.g., as c(vx, vy) = (cvx, cvy).

b. Compute the magnitude of a scalar multiple cv using IIcvII = IcIv. Compute the direction of cv knowing that when IcIv =/o, the direction of cv is either along v (for c>0) or against v (for c<0)

Perform operations on matrices and use matrices in applications

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.

7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.

8. (+) Add, subtract, and multiply matrices of appropriate dimensions.

9. (+) Understand that, unlike multiplications of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.

10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

11. (+) Multiply a vector (regarded as a matrix with one column) by matrix suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

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Seeing the Structure of Expressions

Write expressions in equivalent forms to solve problems

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

c. Use the properties of exponents to transform expressions for exponential functions. For example the expression 1.15^t can be written as $(1.15^{\frac{1}{12}})^{12t} » 1.01212^t$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.

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Understand the relationship between zeros and factors of polynomials

2. Know and apply the Remainder Theorem: For a polynomial p(x) and a number a, the remainder on division by x – a is p(a), so p(a) = 0 if and only if (x – a) is a factor of p(x).

3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

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8. (+) Represent a system of linear equations as a single matrix equation in a vector variable.

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2. Write arithmetic and geometric sequences both recursively and with an explicit formula; use them to model situations, and translate between the two forms.

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Trigonometric Functions

Extend the domain of trigonometric functions using the unit circle

3. (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for π/3, π/4, and π/6,

and use the unit circle to express the values of sine, cosines, and tangent for *x,* π + *x,* and 2π – *x* in terms of their values for *x,* where *x* is any real number.

4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Model periodic phenomena with trigonometric functions

6. (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

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Prove and apply trigonometric identities

9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Expressing Geometric Properties with Equations

Translate between the geometric description and the equation for a conic section

3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

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Prove and apply trigonometric identities

9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

WWSU Computational Fluency Map

Grade Level	Facts (Recall from memory)	Procedural Skills
Kindergarten	Add and subtract within 5 Rote count by ones and tens to 100	Decompose numbers to 10 into pairs One to one correspondence to 20 with moveable objects
First	Addition and subtraction within 10 mentally add or subtract 10 from a 2- digit number Rote count to 120 starting at any number	Add within 100 (2 digit and 1 digit) Add a 2 digit number + a multiple of 10 (42 + 20)
Second	Add and subtract within 20 with mental strategies Commit to memory all sums of two one-digit numbers Mentally add / sub 10 or 100 from numbers 100-900 Skip count by 5s, 10s, 100s to 1000 (forward)	Add / sub within 100 Add up to four 2-digit numbers Expanded form for 3-digit numbers (326=300+20+6)
Third	Commit to memory all products of two one-digit numbers Multiply a single digit number times 10	Add / sub within 1000 Expanded form for 4-digit numbers (6326=6000+300+20+6) Multiply and divide within 100 Decompose 8 x 7 = (8 x 5) + (8 x 2) Multiply a 1-digit number by a multiple of 10 with numbers 10-90 Generate equivalent fractions (denominators 2,3,4,6,8) Express whole numbers as fractions and recognize fractions that are equivalent to whole numbers
Fourth	Recognize equivalent fractions (denominators 2,3,4,5,6,8,10,12,100) Use decimal notation for fractions with denominators of 10 or 100	All factor pairs for numbers 1-100 Fluently add / sub multi-digit whole numbers using an algorithm Multiply a 4-digit by 1-digit Multiply 2-digit by 2-digit Expanded form of 3,428 = (3 x 1000) + (4 x 100) + (2 x 10) + (8 x 1) Divide 4-digit by 1-digit Generate equivalent fractions (denominators 2,3,4,5,6,8,10,12,100) a/b as sum of 1/b fractions (3/5 = 1/5 + 1/5 + 1/5) Add / sub fractions with the same denominator Add / sub mixed numbers with the same denominator Multiply a fraction by a whole number (4 x 1/2 = 1/2+1/2+1/2+1/2) Add 10ths and 100ths in fraction form (by making like denominators)

Fifth	Recognize equivalent forms for fraction, decimal, and percent (denominators 2, 3, 4, 5, 8, 10, 100) Use whole number exponents to denote powers of 10 (10^2=100) Use decimal notation for fractions with denominators of 1000	Write numbers in expanded form - e.g., 347.392 = (3 × 100) + (4 × 10) + (7 × 1) + (3 × 1/10) + (9 × 1/100) + (2 × 1/1000) Multiply multi-digit whole numbers with an algorithm Divide 4-digit by 2-digit Add, sub, mult, div decimals to hundredths Add / sub fractions with unlike denominators (including mixed numbers) Interpret a fraction as division (numerator ÷ denominator) Multiply fraction or whole number by a fraction Divide unit fractions by whole number and whole numbers by unit fractions
Sixth		Find a percent of a quantity as a rate per 100 Divide a fraction by a fraction Fluently divide multi-digit numbers using the standard algorithm Fluently add, sub, mult, divide multi-digit decimals using the standard algorithm Find the greatest common factor and least common multiple of two whole numbers less than 100 Evaluate whole number exponents (4^3 = 4x4x4= 64) Apply order of operations to solve equations

Progressions for the Common Core State Standards in Mathematics (draft)

23 June 2012

K–6, Geometry

Overview

Like core knowledge of number, core geometrical knowledge appears to be a universal capability of the human mind. Geometric and spatial thinking are important in and of themselves, because they connect mathematics with the physical world, and play an important role in modeling phenomena whose origins are not necessarily physical, for example, as networks or graphs. They are also important because they support the development of number and arithmetic concepts and skills. Thus, geometry is essential for all grade levels for many reasons: its mathematical content, its roles in physical sciences, engineering, and many other subjects, and its strong aesthetic connections.

This progression discusses the most important goals for elementary geometry according to three categories.

- Geometric shapes, their components (e.g., sides, angles, faces), their properties, and their categorization based on those properties.
- Composing and decomposing geometric shapes.
- Spatial relations and spatial structuring.

Geometric shapes, components, and properties. Students develop through a series of levels of geometric and spatial thinking. As with all of the domains discussed in the Progressions, this development depends on instructional experiences. Initially, students cannot reliably distinguish between examples and nonexamples of categories of shapes, such as triangles, rectangles, and squares.• With experience, they progress to the next level of thinking, recognizing shapes in ways that are visual or syncretic (a fusion of differing systems). At this level, students can recognize shapes as wholes, but cannot form mathematically-constrained mental images of them. A given figure is a rectangle, for example, because "it looks like a door." They do not explicitly think about the components or about the defining attributes, or properties, of shapes. Students then move to a descriptive level in which they can think about the components of shapes, such as triangles having three sides. For example, kindergartners can decide whether all of the sides of a shape

• In formal mathematics, a geometric shape is a boundary of a region, e.g., "circle" is the boundary of a disk. This distinction is not expected in elementary school.

are straight and they can count the sides. They also can discuss if the shape is closed• and thus convince themselves that a three-sided shape is a triangle even if it is "very skinny" (e.g., an isosceles triangle with large obtuse angle).

At the analytic level, students recognize and characterize shapes by their *properties*.[1] For instance, a student might think of a square as a figure that has four equal sides and four right angles. Different components of shapes are the focus at different grades, for instance, second graders measure lengths and fourth graders measure angles (see the Geometric Measurement Progression). Students find that some combinations of properties signal certain classes of figures and some do not; thus the seeds of geometric implication are planted. However, only at the next level, abstraction, do students see relationships between classes of figures (e.g., understand that a square is a rectangle because it has all the properties of rectangles).• Competence at this level affords the learning of higher-level geometry, including deductive arguments and proof.

Thus, learning geometry cannot progress in the same way as learning number, where the size of the numbers is gradually increased and new kinds of numbers are considered later. In learning about shapes, it is important to vary the examples in many ways so that students do not learn limited concepts that they must later unlearn. From Kindergarten on, students experience all of the properties of shapes that they will study in Grades K–7, recognizing and working with these properties in increasingly sophisticated ways. The Standards describe particular aspects on which students at that grade level work systematically, deeply, and extensively, building on related experiences in previous years.

Composing and decomposing. As with their learning of shapes, components, and properties, students follow a progression to learn about the composition and decomposition of shapes. Initially, they lack competence in composing geometric shapes. With experience, they gain abilities to combine shapes into pictures–first, through trial and error, then gradually using attributes. Finally, they are able to synthesize combinations of shapes into new shapes.•

Students compose new shapes by putting two or more shapes together and discuss the shapes involved as the parts and the totals. They decompose shapes in two ways. They take away a part by covering the total with a part (for example, covering the "top" of a triangle with a smaller triangle to make a trapezoid). And they take shapes apart by building a copy beside the original shape to see what shapes that shape can be decomposed into (initially, they may need to make the decomposition on top of the total shape). With

[1] In this progression, the term "property" is reserved for those attributes that indicate a relationship between components of shapes. Thus, "having parallel sides" or "having all sides of equal lengths" are properties. "Attributes" and "features" are used interchangeably to indicate any characteristic of a shape, including properties, and other defining characteristics (e.g., straight sides) and nondefining characteristics (e.g., "right-side up").

• A shape with straight sides is closed if exactly two sides meet at every vertex, every side meets exactly two other sides, and no sides cross each other.

Levels of geometric thinking

Visual/syncretic. Students recognize shapes, e.g., a rectangle "looks like a door."

Descriptive. Students perceive properties of shapes, e.g., a rectangle has four sides, all its sides are straight, opposite sides have equal length.

Analytic. Students characterize shapes by their properties, e.g., a rectangle has opposite sides of equal length and four right angles.

Abstract. Students understand that a rectangle is a parallelogram because it has all the properties of parallelograms.

• Note that in the U.S., that the term "trapezoid" may have two different meanings. In their study *The Classification of Quadrilaterals* (Information Age Publishing, 2008), Usiskin et al. call these the exclusive and inclusive definitions:

T(E): a trapezoid is a quadrilateral with exactly one pair of parallel sides

T(I): a trapezoid is a quadrilateral with at least one pair of parallel sides.

These different meanings result in different classifications at the analytic level. According to T(E), a parallelogram is not a trapezoid; according to T(I), a parallelogram is a trapezoid.

Both definitions are legitimate. However, Usiskin et al. conclude, "The preponderance of advantages to the inclusive definition of trapezoid has caused all the articles we could find on the subject, and most college-bound geometry books, to favor the inclusive definition."

• **A note about research** The ability to describe, use, and visualize the effects of composing and decomposing geometric regions is significant in that the concepts and actions of creating and then iterating units and higher-order units in the context of constructing patterns, measuring, and computing are established bases for mathematical understanding and analysis. Additionally, there is suggestive evidence that this type of composition corresponds with, and may support, children's ability to compose and decompose numbers.

experience, students are able to use a composed shape as a new unit in making other shapes. Grade 1 students make and use such a unit of units (for example, making a square or a rectangle from two identical right triangles, then making pictures or patterns with such squares or rectangles). Grade 2 students make and use three levels of units (making an isosceles triangle from two 1″ by 2″ right triangles, then making a rhombus from two of such isosceles triangles, and then using such a rhombus with other shapes to make a picture or a pattern). Grade 2 students also compose with two such units of units (for example, making adjacent strips from a shorter parallelograms made from a 1″ by 2″ rectangle and two right triangles and a longer parallelogram made from a 1″ by 3″ parallelogram and the same two right triangles). Grade 1 students also rearrange a composite shape to make a related shape, for example, they change a 1″ by 2″ rectangle made from two right triangles into an isosceles triangle by flipping one right triangle. They explore such rearrangements of the two right triangles more systematically by matching the short right angle side (a tall isosceles triangle and a parallelogram with a "little slant"), then the long right angle sides (a short isosceles triangle and a parallelogram with a "long slant"). Grade 2 students rearrange more complex shapes, for example, changing a parallelogram made from a rectangle and two right triangles into a trapezoid by flipping one of the right triangles to make a longer and a shorter parallel side.

Composing and decomposing requires and thus builds experience with properties such as having equal lengths or equal angles.

Spatial structuring and spatial relations. Early composition and decomposition of shape is a foundation for spatial structuring, an important case of geometric composition and decomposition. Students need to conceptually structure an array to understand two-dimensional objects and sets of such objects in two-dimensional space as truly two-dimensional. Such spatial structuring is the mental operation of constructing an organization or form for an object or set of objects in space, a form of abstraction, the process of selecting, coordinating, unifying, and registering in memory a set of mental objects and actions. Spatial structuring builds on previous shape composition, because it takes previously abstracted items as content and integrates them to form new structures. For two-dimensional arrays, students must see a composite of squares (iterated units) and as a composite of rows or columns (units of units). Such spatial structuring precedes meaningful mathematical use of the structures, including multiplication and, later, area, volume, and the coordinate plane. Spatial relations such as above/below and right/left are understood within such spatial structures. These understandings begin informally, later becoming more formal.

The ability to structure a two-dimensional rectangular region into rows and columns of squares requires extended experiences with shapes derived from squares (e.g., squares, rectangles, and right triangles) and with arrays of contiguous squares that form patterns.

Development of this ability benefits from experience with compositions, decompositions, and iterations of the two, but it requires extensive experience with arrays.

Students make pictures from shapes whose sides or points touch, and they fill in outline puzzles. These gradually become more elaborate, and students build mental visualizations that enable them to move from trial and error rotating of a shape to planning the orientation and moving the shape as it moves toward the target location. Rows and columns are important units of units within square arrays for the initial study of area, and squares of 1 by 1, 1 by 10, and 10 by 10 are the units, units of units, and units of units of units used in area models of two-digit multiplication in Grade 4. Layers of three-dimensional shapes are central for studying volume in Grade 5. Each layer of a right rectangular prism can also be structured in rows and columns, such layers can also be viewed as units of units of units. That is, as 1000 is a unit (one thousand) of units (one hundred) of units (tens) of units (singletons), a right rectangular prism can be considered a unit (solid, or three-dimensional array) of units (layers) of units (rows) of units (unit cubes).

Summary. The Standards for Kindergarten, Grade 1, and Grade 2 focus on three major aspects of geometry. Students build understandings of shapes and their properties, becoming able to do and discuss increasingly elaborate compositions, decompositions, and iterations of the two, as well as spatial structures and relations. In Grade 2, students begin the formal study of measure, learning to use units of length and use and understand rulers. Measurement of angles and parallelism are a focus in Grades 3, 4, and 5. At Grade 3, students begin to consider relationships of shape categories, considering two levels of subcategories (e.g., rectangles are parallelograms and squares are rectangles). They complete this categorization in Grade 5 with all necessary levels of categories and with the understanding that any property of a category also applies to all shapes in any of its subcategories. They understand that some categories overlap (e.g., not all parallelograms are rectangles) and some are disjoint (e.g., no square is a triangle), and they connect these with their understanding of categories and subcategories. Spatial structuring for two- and three-dimensional regions is used to understand what it means to measure area and volume of the simplest shapes in those dimensions: rectangles at Grade 3 and right rectangular prisms at Grade 5 (see the Geometric Measurement Progression).

K.G.4 Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).

Kindergarden

Understanding and describing shapes and space is one of the two critical areas of Kindergarten mathematics. Students develop geometric concepts and spatial reasoning from experience with two perspectives on space: the shapes of objects and the relative positions of objects.

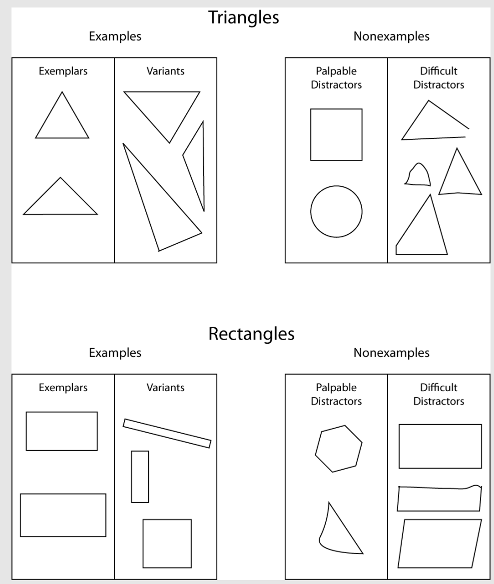
In the domain of shape, students learn to match two-dimensional shapes even when the shapes have different orientations.[K.G.4] They learn to name shapes such as circles, triangles, and squares, whose names occur in everyday language, and distinguish them from nonexamples of these categories, often based initially on visual prototypes. For example, they can distinguish the most typical examples of triangles from the obvious nonexamples.

From experiences with varied examples of these shapes (e.g., the variants shown in the margin), students extend their initial intuitions to increasingly comprehensive and accurate intuitive concept images of each shape category.• These richer concept images support students' ability to perceive a variety of shapes in their environments and describe these shapes in their own words.[MP7] This includes recognizing and informally naming three-dimensional shapes, e.g., "balls," "boxes," "cans." Such learning might also occur in the context of solving problems that arise in construction of block buildings and in drawing pictures, simple maps, and so forth.

Students then refine their informal language by learning mathematical concepts and vocabulary so as to increasingly describe their physical world from geometric perspectives, e.g., shape, orientation, spatial relations (MP4). They increase their knowledge of a variety of shapes, including circles, triangles, squares, rectangles, and special cases of other shapes such as regular hexagons, and trapezoids with unequal bases and non-parallel sides of equal length.[K.G.1] • They learn to sort shapes according to these categories.[MP7] The need to explain their decisions about shape names or classifications prompts students to attend to and describe certain features of the shapes.[K.G.4] That is, concept images and names they have learned for the shapes are the raw material from which they can abstract common features.[MP2] This also supports their learning to represent shapes informally with drawings and by building them from components (e.g., manipulatives such as sticks).[K.G.5] With repeated experiences such as these, students become more precise (MP6). They begin to attend to attributes, such as being a triangle, square, or rectangle, and being *closed* figures with *straight* sides. Similarly, they attend to the lengths of sides and, in simple situations, the size of angles when comparing shapes.

Students also begin to name and describe three-dimensional shapes with mathematical vocabulary, such as "sphere," "cube," "cylinder," and "cone."[K.G.1] They identify faces of three-dimensional shapes as two-dimensional geometric figures[K.G.4] and explicitly identify shapes as two-dimensional ("flat" or lying in a plane) or three-dimensional

[K.G.4] Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).



Exemplars are the typical visual prototypes of the shape category.

Variants are other examples of the shape category.

Palpable distractors are nonexamples with little or no overall resemblance to the exemplars.

Difficult distractors are visually similar to examples but lack at least one defining attribute.

• Tall and Vinner describe *concept image* as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built over the years through experiences of all kinds, changing as the individual meets new stimuli and matures." (See "Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity," *Educational Studies in Mathematics, 12*, pp. 151–169.) This term was formulated by Shlomo Vinner in 1980.

[MP7] Mathematically proficient students look closely to discern a pattern or structure.

[K.G.1] Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above*, *below*, *beside*, *in front of*, *behind*, and *next to*.

• If the exclusive definition of trapezoid is used (see p. 3), such trapezoids would be called isosceles trapezoids.

[MP7] Young students, for example, . . . may sort a collection of shapes according to how many sides the shapes have.

[MP2] Mathematically proficient students have the ability to abstract a given situation.

[K.G.5] Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes.

("solid").K.G.3

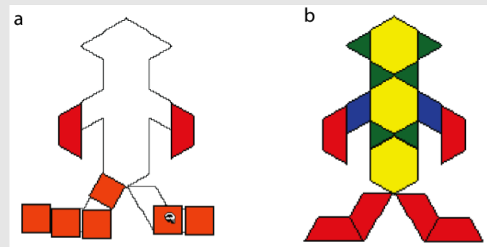
A second important area for kindergartners is the composition of geometric figures. Students not only build shapes from components, but also compose shapes to build pictures and designs. Initially lacking competence in composing geometric shapes, they gain abilities to combine shapes–first by trial and error and gradually by considering components–into pictures. At first, side length is the only component considered. Later experience brings an intuitive appreciation of angle size.

Students combine two-dimensional shapes and solve problems such as deciding which piece will fit into a space in a puzzle, intuitively using geometric motions (slides, flips, and turns, the informal names for translations, reflections, and rotations, respectively). They can construct their own outline puzzles and exchange them, solving each other's.

Finally, in the domain of spatial reasoning, students discuss not only shape and orientation, but also the relative positions of objects, using terms such as "above," "below," "next to," "behind," "in front of," and "beside."K.G.1 They use these spatial reasoning competencies, along with their growing knowledge of three-dimensional shapes and their ability to compose them, to model objects in their environment, e.g., building a simple representation of the classroom using unit blocks and/or other solids (MP4).

K.G.3 Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").

Combining shapes to build pictures



Students first use trial and error (part a) and gradually consider components (part b).

Grade 1

In Grade 1, students reason about shapes. They describe and classify shapes, including drawings, manipulatives, and physical-world objects, in terms of their geometric *attributes*. That is, based on early work recognizing, naming, sorting, and building shapes from components, they describe in their own words why a shape belongs to a given category, such as squares, triangles, circles, rectangles, rhombuses, (regular) hexagons, and trapezoids (with bases of different lengths and nonparallel sides of the same length). In doing so, they differentiate between geometrically defining attributes (e.g., "hexagons have six straight sides") and nondefining attributes (e.g., color, overall size, or orientation).[1.G.1] For example, they might say of this shape, "This has to go with the squares, because all four sides are the same, and these are square corners. It doesn't matter which way it's turned" (MP3, MP7). They explain why the variants shown earlier (p. 6) are members of familiar shape categories and why the difficult distractors are not, and they draw examples and nonexamples of the shape categories. Students learn to sort shapes accurately and exhaustively based on these attributes, describing the similarities and differences of these familiar shapes and shape categories (MP7, MP8).

From the early beginnings of informally matching shapes and solving simple shape puzzles, students learn to intentionally compose and decompose plane and solid figures (e.g., putting two congruent isosceles triangles together with the explicit purpose of making a rhombus),[1.G.2] building understanding of part-whole relationships as well as the properties of the original and composite shapes. In this way, they learn to perceive a combination of shapes as a single new shape (e.g., recognizing that two isosceles triangles can be combined to make a rhombus, and simultaneously seeing the rhombus and the two triangles). Thus, they develop competencies that include solving shape puzzles and constructing designs with shapes, creating and maintaining a shape as a unit, and combining shapes to create composite shapes that are conceptualized as independent entities (MP2). They then learn to substitute one composite shape for another congruent composite composed of different parts.

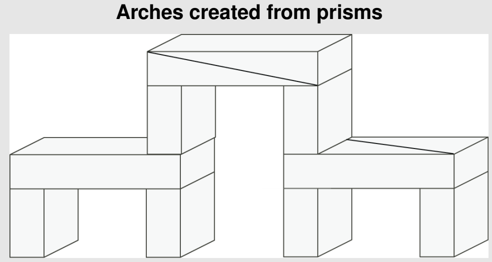
Students build these competencies, often more slowly, in the domain of three-dimensional shapes. For example, students may intentionally combine two right triangular prisms to create a right rectangular prism, and recognize that each triangular prism is half of the rectangular prism.[1.G.3] They also show recognition of the composite shape of "arch." (Note that the process of combining shapes to create a composite shape is much like combining 10 ones to make 1 ten.) Even simple compositions, such as building a floor or wall of rectangular prisms, build a foundation for later mathematics.

As students combine shapes, they continue to develop their sophistication in describing geometric attributes and properties and determining how shapes are alike and different, building founda-

[1.G.1] Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size) ; build and draw shapes to possess defining attributes.

[1.G.2] Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.[2]

Arches created from prisms



Right rectangular prisms are composed with prisms with right triangle bases. Note that the dimensions of the triangular prism on the top arch differ from the dimensions of that on the right.

[1.G.3] Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

tions for measurement and initial understandings of properties such as congruence and symmetry. Students can learn to use their intuitive understandings of measurement, congruence, and symmetry to guide their work on tasks such as solving puzzles and making simple origami constructions by folding paper to make a given two- or three-dimensional shape (MP1).•

• For example, students might fold a square of paper once to make a triangle or nonsquare rectangle. For examples of other simple two- and three-dimensional origami constructions, see `http://www.origami-instructions.com/simple-origami.html`.

Grade 2

Students learn to name and describe the defining attributes of categories of two-dimensional shapes, including circles, triangles, squares, rectangles, rhombuses, trapezoids, and the general category of quadrilateral. They describe pentagons, hexagons, septagons, octagons, and other polygons by the number of sides, for example, describing a septagon as either a "seven-gon" or simply "seven-sided shape" (MP2).[2.G.1] Because they have developed both verbal descriptions of these categories and their defining attributes and a rich store of associated mental images, they are able to draw shapes with specified attributes, such as a shape with five sides or a shape with six angles.[2.G.1] They can represent these shapes' attributes accurately (within the constraints of fine motor skills). They use length to identify the properties of shapes (e.g., a specific figure is a rhombus because all four of its sides have equal length). They recognize right angles, and can explain the distinction between a rectangle and a parallelogram without right angles and with sides of different lengths (sometimes called a "rhomboid").

Students learn to combine their composition and decomposition competencies to build and operate on composite units (units of units), intentionally substituting arrangements or composites of smaller shapes or substituting several larger shapes for many smaller shapes, using geometric knowledge and spatial reasoning to develop foundations for area, fraction, and proportion. For example, they build the same shape from different parts, e.g., making with pattern blocks, a regular hexagon from two trapezoids, three rhombuses, or six equilateral triangles. They recognize that the hexagonal faces of these constructions have equal area, that each trapezoid has half of that area, and each rhombus has a third of that area.[2.G.3]

This example emphasizes the fraction concepts that are developed; students can build and recognize more difficult composite shapes and solve puzzles with numerous pieces. For example, a tangram is a special set of 7 shapes which compose an isosceles right triangle. The tangram pieces can be used to make many different configurations and tangram puzzles are often posed by showing pictures of these configurations as silhouettes or outlines. These pictures often are made more difficult by orienting the shapes so that the sides of right angles are not parallel to the edges of the page on which they are displayed. Such pictures often do not show a grid that shows the composing shapes and are generally not preceded by analysis of the composing shapes.

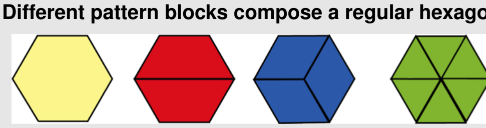
Students also explore decompositions of shapes into regions that are congruent or have equal area.[2.G.3] For example, two squares can be partitioned into fourths in different ways. Any of these fourths represents an equal share of the shape (e.g., "the same amount of cake") even though they have different shapes.

Another type of composition and decomposition is essential to students' mathematical development—*spatial structuring*. Students

[2.G.1] Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.[3] Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.

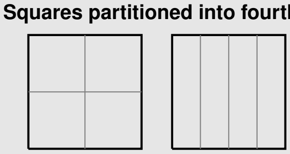
[2.G.1] Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.[4] Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.

Different pattern blocks compose a regular hexagon



[2.G.3] Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

Squares partitioned into fourths



These different partitions of a square afford the opportunity for students to identify correspondences between the differently-shaped fourths (MP.1), and to explain how one of the fourths on the left can be transformed into one of the fourths on the right (MP.7).

[2.G.3] Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

need to conceptually structure an array to understand two-dimensional regions as truly two-dimensional. This involves more learning than is sometimes assumed. Students need to understand how a rectangle can be tiled with squares lined up in rows and columns.[2.G.2] At the lowest level of thinking, students draw or place shapes inside the rectangle, but do not cover the entire region. Only at the later levels do all the squares align vertically and horizontally, as the students learn to compose this two-dimensional shape as a collection of rows of squares and as a collection of columns of squares (MP7).

Spatial structuring is thus the mental operation of constructing an organization or form for an object or set of objects in space, a form of abstraction, the process of selecting, coordinating, unifying, and registering in memory a set of mental objects and actions. Spatial structuring builds on previous shape composition, because previously abstracted items (e.g., squares, including composites made up of squares) are used as the content of new mental structures. Students learn to see an object such as a row in two ways: as a composite of multiple squares and as a single entity, a row (a unit of units). Using rows or columns to cover a rectangular region is, at least implicitly, a composition of units. At first, students might tile a rectangle with identical squares or draw such arrays and then count the number of squares one-by-one. In the lowest levels of the progression, they may even lose count of or double-count some squares. As the mental structuring process helps them organize their counting, they become more systematic, using the array structure to guide the quantification. Eventually, they begin to use repeated addition of the number in each row or each column. Such spatial structuring precedes meaningful mathematical use of the structures, including multiplication and, later, area, volume, and the coordinate plane.

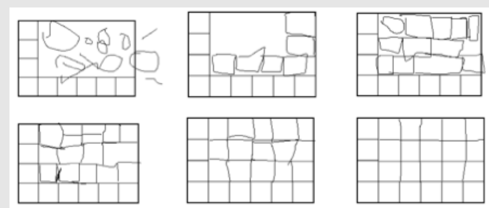
Foundational activities, such as forming arrays by tiling a rectangle with identical squares (as in building a floor or wall from blocks) should have developed students' basic spatial structuring competencies before second grade—if not, teachers should ensure that their students learn these skills. Spatial structuring can be further developed with several activities with grids. Games such as "battleship" can be useful in this regard.

Another useful type of instructional activity is copying and creating designs on grids. Students can copy designs drawn on grid paper by placing manipulative squares and right triangles onto other copies of the grid. They can also create their own designs, draw their creations on grid paper, and exchange them, copying each others' designs.

Another, more complex, activity designing tessellations by iterating a "core square." Students design a unit composed of smaller units: a core square composed of a 2 by 2 array of squares filled with square or right triangular regions. They then create the tessellation ("quilt") by iterating that core in the plane. This builds spatial structuring because students are iterating "units of units" and reflecting on the resulting structures. Computer software can

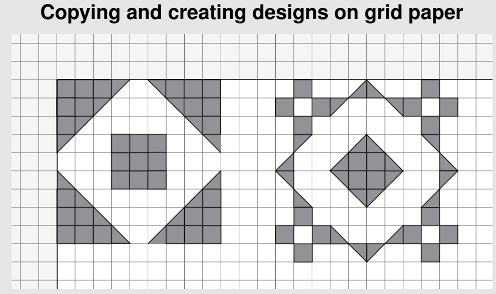
[2.G.2] Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

Levels of thinking in spatial structuring



Levels of thinking portrayed by different students as they attempted to complete a drawing of an array of squares, given one column and row. This was an assessment, not an instructional task.

Copying and creating designs on grid paper

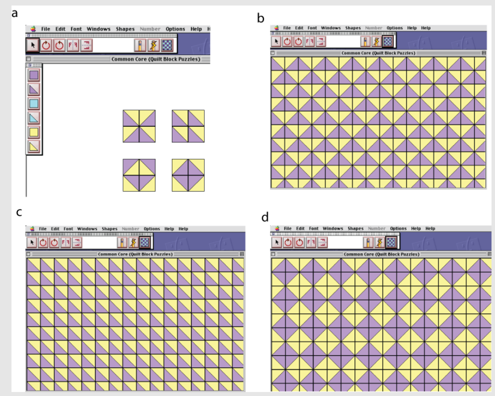


Students can copy designs such as these, using only squares (all of the same size) and isosceles right triangles (half of the square) as manipulatives, creating their copies on paper with grid squares of the same size as the manipulative square.

aid in this iteration.

These various types of composition and decomposition experiences simultaneously develop students' visualization skills, including recognizing, applying, and anticipating (MP1) the effects of applying rigid motions (slides, flips, and turns) to two-dimensional shapes.

"Core squares" iterated to make a tessellation



In the software environment illustrated above (Pattern Blocks and Mini-Quilts software), students need to be explicitly aware of the transformations they are using in order to use slide, flip, and turn tools. At any time, they can tessellate any one of the core squares using the "quilt" tool indicated by the rightmost icon. Part a shows four different core squares. The upper left core square produces the tessellation in part b. Parts c and d are produced, respectively, by the upper right and lower right core squares. Interesting discussions result when the class asks which designs are mathematically different (e.g., should a rotation or flip of the core be counted as "different"s?).

Grade 3

Students analyze, compare, and classify two-dimensional shapes by their properties (see the footnote on p. 3).[3.G.1] They explicitly relate and combine these classifications. Because they have built a firm foundation of several shape categories, these categories can be the raw material for thinking about the relationships between classes. For example, students can form larger, superordinate, categories, such as the class of all shapes with four sides, or quadrilaterals, and recognize that it includes other categories, such as squares, rectangles, rhombuses, parallelograms, and trapezoids. They also recognize that there are quadrilaterals that are not in any of those subcategories. A description of these categories of quadrilaterals is illustrated in the margin. The Standards do not require that such representations be constructed by Grade 3 students, but they should be able to draw examples of quadrilaterals that are not in the subcategories.

Similarly, students learn to draw shapes with prespecified attributes, without making a priori assumptions regarding their classification.[MP1] For example, they could solve the problem of making a shape with two long sides of the same length and two short sides of the same length that is not a rectangle.

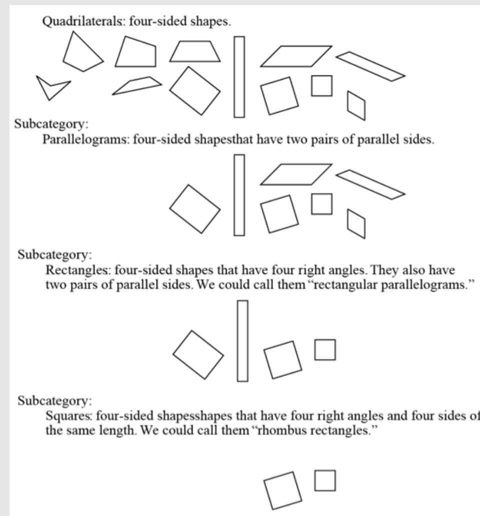
Students investigate, describe, and reason about decomposing and composing polygons to make other polygons. Problems such as finding all the possible different compositions of a set of shapes involve geometric problem solving and notions of congruence and symmetry (MP7). They also involve the practices of making and testing conjectures (MP1), and convincing others that conjectures are correct (or not) (MP3). Such problems can be posed even for sets of simple shapes such as tetrominoes, four squares arranged to form a shape so that every square shares at least one side and sides coincide or share only a vertex.

More advanced paper-folding (origami) tasks afford the same mathematical practices of seeing and using structure, conjecturing, and justifying conjectures. Paper folding can also illustrate many geometric concepts. For example, folding a piece of paper creates a line segment. Folding a square of paper twice, horizontal edge to horizontal edge, then vertical edge to vertical edge, creates a right angle, which can be unfolded to show four right angles. Students can be challenged to find ways to fold paper into rectangles or squares and to explain why the shapes belong in those categories.

Students also develop more competence in the composition and decomposition of rectangular regions, that is, spatially structuring rectangular arrays. They learn to partition a rectangle into identical squares[3.G.2] by anticipating the final structure and thus forming the array by drawing rows and columns (see the bottom right example on p. 11; some students may still need work building or drawing squares inside the rectangle first). They count by the number of columns or rows, or use multiplication to determine the number of

3.G.1 Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.

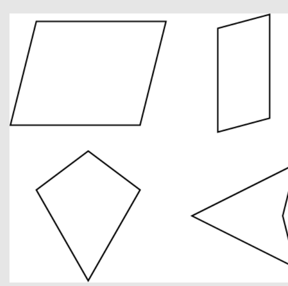
Quadrilaterals and some special kinds of quadrilaterals



The representations above might be used by teachers in class. Note that the left-most four shapes in the first section at the top left have four sides but do not have properties that would place them in any of the other categories shown (parallelograms, rectangles, squares).

MP1 Students . . . analyze givens, constraints, relationships, and goals.

Quadrilaterals that are not rectangles



These quadrilaterals have two pairs of sides of the same length but are not rectangles. A kite is on lower left and a deltoid is at lower right.

3.G.2 Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole.

squares in the array. They also learn to rotate these arrays physically and mentally to view them as composed of smaller arrays, allowing illustrations of properties of multiplication (e.g., the commutative property and the distributive property).

Grade 4

Students describe, analyze, compare, and classify two-dimensional shapes by their properties (see the footnote on p. 3), including explicit use of angle sizes[4.G.1] and the related geometric properties of perpendicularity and parallelism.[4.G.2] They can identify these properties in two-dimensional figures. They can use side length to classify triangles as equilateral, equiangular, isosceles, or scalene; and can use angle size to classify them as acute, right, or obtuse. They then learn to cross-classify, for example, naming a shape as a right isosceles triangle. Thus, students develop explicit awareness of and vocabulary for many concepts they have been developing, including points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Such mathematical terms are useful in communicating geometric ideas, but more important is that constructing examples of these concepts, such as drawing angles and triangles that are acute, obtuse, and right,[4.G.1] help students form richer concept images connected to verbal definitions. That is, students have more complete and accurate mental images and associated vocabulary for geometric ideas (e.g., they understand that angles can be larger than 90° and their concept images for angles include many images of such obtuse angles). Similarly, students see points and lines as abstract objects: Lines are infinite in extent and points have location but no dimension. Grids are made of points and lines and do not end at the edge of the paper.

Students also learn to apply these concepts in varied contexts (MP4). For example, they learn to represent angles that occur in various contexts as two rays, explicitly including the reference line, e.g., a horizontal or vertical line when considering slope or a "line of sight" in turn contexts. They understand the size of the angle as a rotation of a ray on the reference line to a line depicting slope or as the "line of sight" in computer environments. Students might solve problems of drawing shapes with turtle geometry.• Analyzing the shapes in order to construct them (MP1) requires students to explicitly formulate their ideas about the shapes (MP4, MP6). For instance, what series of commands would produce a square? How many degrees would the turtle turn? What is the measure of the resulting angle? What would be the commands for an equilateral triangle? How many degrees would the turtle turn? What is the measure of the resulting angle? Such experiences help students connect what are often initially isolated ideas about the concept of angle.

Students might explore line segments, lengths, perpendicularity, and parallelism on different types of grids, such as rectangular and

4.G.1 Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

4.G.2 Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

4.G.1 Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

• The computer programming language Logo has a pointer, often a icon of a turtle, that draws representations of points, line segments, and shapes, with commands such as "forward 100" and "right 120."

triangular (isometric) grids (MP1, MP2).[4.G.2, 4.G.3] Can you find a non-rectangular parallelogram on a rectangular grid? Can you find a rectangle on a triangular grid? Given a segment on a rectangular grid that is not parallel to a grid line, draw a parallel segment of the same length with a given endpoint. Given a half of a figure and a line of symmetry, can you accurately draw the other half to create a symmetric figure?

Students also learn to reason about these concepts. For example, in "guess my rule" activities, they may be shown two sets of shapes and asked where a new shape belongs (MP1, MP2).[4.G.2]

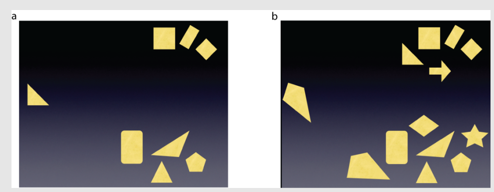
In an interdisciplinary lesson (that includes science and engineering ideas as well as items from mathematics), students might encounter another property that all triangles have: rigidity. If four fingers (both thumbs and index fingers) form a shape (keeping the fingers all straight), the shape of that quadrilateral can be easily changed by changing the angles. However, using three fingers (e.g., a thumb on one hand and the index and third finger of the other hand), students can see that the shape is fixed by the side lengths. Triangle rigidity explains why this shape is found so frequently in bridge, high-wire towers, amusement park rides, and other constructions where stability is sought.

4.G.2 Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

4.G.3 Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

4.G.2 Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

Guess My Rule



Students can be shown the two groups of shapes in part a and asked "Where does the shape on the left belong?" They might surmise that it belongs with the other triangles at the bottom. When the teacher moves it to the top, students must search for a different rule that fits all the cases.

Later (part b), students may induce the rule: "Shapes with at least one right angle are at the top." Students with rich visual images of right angles and good visualization skills would conclude that the shape at the left (even though it looks vaguely like another one already at the bottom) has one right angle, thus belongs at the top.

Grade 5

By the end of Grade 5, competencies in shape composition and decomposition, and especially the special case of spatial structuring of rectangular arrays (recall p. 11), should be highly developed (MP7). Students need to develop these competencies because they form a foundation for understanding multiplication, area, volume, and the coordinate plane. To solve area problems, for example, the ability to decompose and compose shapes plays multiple roles. First, students understand that the area of a shape (in square units) is the number of unit squares it takes to cover the shape without gaps or overlaps. They also use decomposition in other ways. For example, to calculate the area of an "L-shaped" region, students might decompose the region into rectangular regions, then decompose each region into an array of unit squares, spatially structuring each array into rows or columns. Students extend their spatial structuring in two ways. They learn to spatially structure in three dimensions; for example, they can decompose a right rectangular prism built from cubes into layers, seeing each layer as an array of cubes. They use this understanding to find the volumes of right rectangular prisms with edges whose lengths are whole numbers as the number of unit cubes that pack the prisms (see the Geometric Measurement Progression). Second, students extend their knowledge of the coordinate plane, understanding the continuous nature of two-dimensional space and the role of fractions in specifying locations in that space.

Thus, spatial structuring underlies coordinates for the plane as well, and students learn both to apply it and to distinguish the objects that are structured. For example, they learn to interpret the components of a rectangular grid structure as line segments or lines (rather than regions) and understand the precision of location that these lines require, rather than treating them as fuzzy boundaries or indicators of intervals. Students learn to reconstruct the levels of counting and quantification that they had already constructed in the domain of discrete objects to the coordination of (at first) two continuous linear measures. That is, they learn to apply their knowledge of number and length to the order and distance relationships of a coordinate grid and to coordinate this across two dimensions.[5.G.1]

Although students can often "locate a point," these understandings are beyond simple skills. For example, initially, students often fail to distinguish between two different ways of viewing the point $(2, 3)$, say, as instructions: "right 2, up 3"; and as the point defined by being a distance 2 from the y-axis and a distance 3 from the x-axis. In these two descriptions the 2 is first associated with the x-axis, then with the y-axis.

They connect ordered pairs of (whole number) coordinates to points on the grid, so that these coordinate pairs constitute numerical objects and ultimately can be operated upon as single mathematical entities. Students solve mathematical and real-world problems using coordinates. For example, they plan to draw a symmetric fig-

[5.G.1] Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., x-axis and x-coordinate, y-axis and y-coordinate).

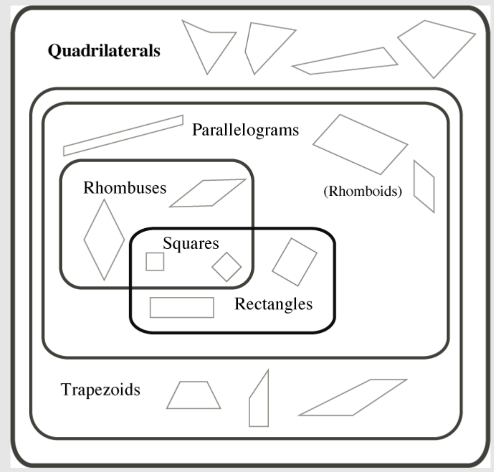
ure using computer software in which students' input coordinates that are then connected by line segments.[5.G.2]

Students learn to analyze and relate categories of two-dimensional and three-dimensional shapes explicitly based on their properties.[5.G.4] Based on analysis of properties, they classify two-dimensional figures in hierarchies. For example, they conclude that all rectangles are parallelograms, because they are all quadrilaterals with two pairs of opposite, parallel, equal-length sides (MP3). In this way, they relate certain categories of shapes as subclasses of other categories.[5.G.3] This leads to understanding propagation of properties; for example, students understand that squares possess all properties of rhombuses and of rectangles. Therefore, if they then show that rhombuses' diagonals are perpendicular bisectors of one another, they infer that squares' diagonals are perpendicular bisectors of one another as well.

[5.G.2] Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.

[5.G.4] Classify two-dimensional figures in a hierarchy based on properties.

Venn diagram showing classification of quadrilaterals



Note that rhomboids are parallelograms that are not rhombuses or rectangles. This example uses the inclusive definition of trapezoid (see p. [pageref "T(E)"]).

[5.G.3] Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category.

Grade 6

Problems involving areas and volumes extend previous work and provide a context for developing and using equations.[6.G.1, 6.G.2] Students' competencies in shape composition and decomposition, especially with spatial structuring of rectangular arrays (recall p. 11), should be highly developed. These competencies form a foundation for understanding multiplication, formulas for area and volume, and the coordinate plane.[6.NS.6, 6.NS.8]

Using the shape composition and decomposition skills acquired in earlier grades, students learn to develop area formulas for parallelograms, then triangles. They learn how to address three different cases for triangles: a height that is a side of a right angle, a height that "lies over the base" and a height that is outside the triangle.[MP.1]

Through such activity, students learn that that any side of a triangle can be considered as a base and the choice of base determines the height (thus, the base is not necessarily horizontal and the height is not always in the interior of the triangle). The ability to view a triangle as part of a parallelogram composed of two copies of that triangle and the understanding that area is additive (see the Geometric Measurement Progression) provides a justification (MP3) for halving the product of the base times the height, helping students guard against the common error of forgetting to take half.

Also building on their knowledge of composition and decomposition, students decompose rectilinear polygons into rectangles, and decompose special quadrilaterals and other polygons into triangles and other shapes, using such decompositions to determine their areas, and justifying and finding relationships among the formulas for the areas of different polygons.

Building on the knowledge of volume (see the Geometric Measurement Progression) and spatial structuring abilities developed in earlier grades, students learn to find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism.[6.G.2 MP.1 MP.4]

Students also analyze and compose and decompose polyhedral solids. They describe the shapes of the faces, as well as the number of faces, edges, and vertices. They make and use drawings of solid shapes and learn that solid shapes have an outer surface as well as an interior. They develop visualization skills connected to their mathematical concepts as they recognize the existence of, and visualize, components of three-dimensional shapes that are not visible from a given viewpoint (MP1). They measure the attributes of these shapes, allowing them to apply area formulas to solve surface area problems (MP7). They solve problems that require them to distinguish between units used to measure volume and units used to measure area (or length). They learn to plan the construction of

6.G.1 Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

6.G.2 Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

6.NS.6 Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

6.NS.8 Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

MP.1 Students . . . try special cases and simpler forms of the original problem in order to gain insight into its solution.

6.G.2 Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

MP.1 explain correspondences

MP.4 write an equation to describe a situation.

complex three-dimensional compositions through the creation of corresponding two-dimensional nets (e.g., through a process of digital fabrication and/or graph paper).[6.G.4] For example, they may design a living quarters (e.g., a space station) consistent with given specifications for surface area and volume (MP2, MP7). In this and many other contexts, students learn to apply these strategies and formulas for areas and volumes to the solution of real-world and mathematical problems.[6.G.1, 6.G.2] These problems include those in which areas or volumes are to be found from lengths or lengths are to be found from volumes or areas and lengths.

Students extend their understanding of properties of two-dimensional shapes to use of coordinate systems.[6.G.3] For example, they may specify coordinates for a polygon with specific properties, justifying the attribution of those properties through reference to relationships among the coordinates (e.g., justifying that a shape is a parallelogram by computing the lengths of its pairs of horizontal and vertical sides).

As a precursor for learning to describe cross-sections of three-dimensional figures,[7.G.3] students use drawings and physical models to learn to identify parallel lines in three-dimensional shapes, as well as lines perpendicular to a plane, lines parallel to a plane, the plane passing through three given points, and the plane perpendicular to a given line at a given point.

[6.G.4] Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

[6.G.1] Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

[6.G.2] Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

[6.G.3] Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.

[7.G.3] Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.

Where the Geometry Progression is Heading

Composition and decomposition of shapes is used throughout geometry from Grade 6 to high school and beyond. Compositions and decompositions of regions continues to be important for solving a wide variety of area problems, including justifications of formulas and solving real world problems that involve complex shapes. Decompositions are often indicated in geometric diagrams by an auxiliary line, and using the strategy of drawing an auxiliary line to solve a problem are part of looking for and making use of structure (MP7). Recognizing the significance of an existing line in a figure is also part of looking for and making use of structure. This may involve identifying the length of an associated line segment, which in turn may rely on students' abilities to identify relationships of line segments and angles in the figure. These abilities become more sophisticated as students gain more experience in geometry. In Grade 7, this experience includes making scale drawings of geometric figures and solving problems involving angle measure, surface area, and volume (which builds on understandings described in the Geometric Measurement Progression as well as the ability to compose and decompose figures).

Progressions for the Common Core State Standards in Mathematics (draft)

20 June 2011

K–3, Categorical Data; Grades 2–5, Measurement Data*

Overview

As students work with data in Grades K–5, they build foundations for their study of statistics and probability in Grades 6 and beyond, and they strengthen and apply what they are learning in arithmetic. Kindergarten work with data uses counting and order relations. First- and second-graders solve addition and subtraction problems in a data context. In Grades 3–5, work with data is closely related to the number line, fraction concepts, fraction arithmetic, and solving problems that involve the four operations. See Table 1 for these and other notable connections between arithmetic and data work in Grades K–5.

As shown in Table 1, the K–5 data standards run along two paths. One path deals with *categorical data* and focuses on bar graphs as a way to represent and analyze such data. Categorical data comes from sorting objects into categories—for example, sorting a jumble of alphabet blocks to form two stacks, a stack for vowels and a stack for consonants. In this case there are two categories (Vowels and Consonants). Students' work with categorical data in early grades will support their later work with bivariate categorical data and two-way tables in eighth grade (this is discussed further at the end of the Categorical Data Progression).

The other path deals with *measurement data*. As the name suggests, measurement data comes from taking measurements. For example, if every child in a class measures the length of his or her hand to the nearest centimeter, then a set of measurement data is obtained. Other ways to generate measurement data might include measuring liquid volumes with graduated cylinders or measuring room temperatures with a thermometer. In each case, the Standards call for students to represent measurement data with a *line plot*.

*These progressions concern Measurement and Data standards related to data. Other MD standards are discussed in the Geometric Measurement Progression.

This is a type of display that positions the data along the appropriate scale, drawn as a number line diagram. These plots have two names in common use, "dot plot" (because each observation is represented as a dot) and "line plot" (because each observation is represented above a number line diagram).

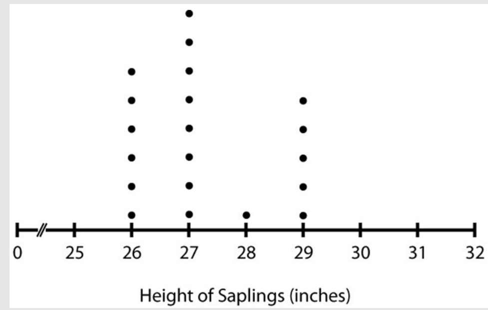
The number line diagram in a line plot corresponds to the scale on the measurement tool used to generate the data. In a context involving measurement of liquid volumes, the scale on a line plot could correspond to the scale etched on a graduated cylinder. In a context involving measurement of temperature, one might imagine a picture in which the scale on the line plot corresponds to the scale printed on a thermometer. In the last two cases, the correspondence may be more obvious when the scale on the line plot is drawn vertically.

Students should understand that the numbers on the scale of a line plot indicate the total number of measurement units from the zero of the scale.

Students need to choose appropriate representations (MP5), labeling axes to clarify the correspondence with the quantities in the situation and specifying units of measure (MP6). Measuring and recording data require attention to precision (MP6). Students should be supported as they learn to construct picture graphs, bar graphs, and line plots. Grid paper should be used for assignments as well as assessments. This may help to minimize errors arising from the need to track across a graph visually to identify values. Also, a template can be superimposed on the grid paper, with blanks provided for the student to write in the graph title, scale labels, category labels, legend, and so on. It might also help if students write relevant numbers on graphs during problem solving.

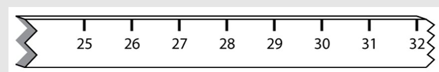
In students' work with data, context is important. As the *Guidelines for Assessment and Instruction in Statistics Education Report* notes, "data are not just numbers, they are numbers with a context. In mathematics, context obscures structure. In data analysis, context provides meaning."• In keeping with this perspective, students should work with data in the context of science, social science, health, and other subjects, always interpreting data plots in terms of the data they represent (MP2).

Example of a line plot



Note that the break in the scale between 0 and 25 indicates that marks between 0 and 25 are not shown.

Example of a scale on a measurement tool



• The *Guidelines for Assessment and Instruction in Statistics Education Report* was published in 2007 by the American Statistical Association, http://www.amstat.org/education/gaise.

Table 1: Some notable connections to K–5 data work

Grade	Standard	Notable Connections
	Categorical data	
K	K.MD.3. Classify objects into given categories, count the number of objects in each category and sort[1] the categories by count. *Limit category counts to be less than or equal to 10.*	• K.CC. Counting to tell the number of objects • K.CC. Comparing numbers
1	1.MD.4. Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.	• 1.OA. Problems involving addition and subtraction ◦ put-together, take-apart, compare ◦ problems that call for addition of three whole numbers
2	2.MD.10. Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph.	• 2.OA. Problems involving addition and subtraction ◦ put-together, take-apart, compare
3	3.MD.3. Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. *For example, draw a bar graph in which each square in the bar graph might represent 5 pets.*	• 3.OA.3. Problems involving multiplication • 3.OA.8 Two-step problems using the four operations • 3.G.1 Categories of shapes
	Measurement data	
2	2.MD.9. Generate measurement data by measuring lengths of several objects to the nearest whole unit, or by making repeated measurements of the same object. Show the measurements by making a line plot, where the horizontal scale is marked off in whole-number units.	• 1.MD.2. Length measurement • 2.MD.6. Number line
3	3.MD.4. Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units–whole numbers, halves, or quarters.	• 3.NF.2. Fractions on a number line
4	4.MD.4. Make a line plot to display a data set of measurements in fractions of a unit $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$. Solve problems involving addition and subtraction of fractions by using information presented in line plots. *For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.*	• 4.NF.3,4. Problems involving fraction arithmetic
5	5.MD.2. Make a line plot to display a data set of measurements in fractions of a unit $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$. Use operations on fractions for this grade to solve problems involving information presented in line plots. *For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.*	• 5.NF.1,2,4,6,7. Problems involving fraction arithmetic

[1] Here, "sort the categories" means "order the categories," i.e., show the categories in order according to their respective counts.

Categorical Data

Kindergarten

Students in Kindergarten classify objects into categories, initially specified by the teacher and perhaps eventually elicited from students. For example, in a science context, the teacher might ask students in the class to sort pictures of various organisms into two piles: organisms with wings and those without wings. Students can then count the number of specimens in each pile.[K.CC.5] Students can use these category counts and their understanding of cardinality to say whether there are more specimens with wings or without wings.[K.CC.6,K.CC.7]

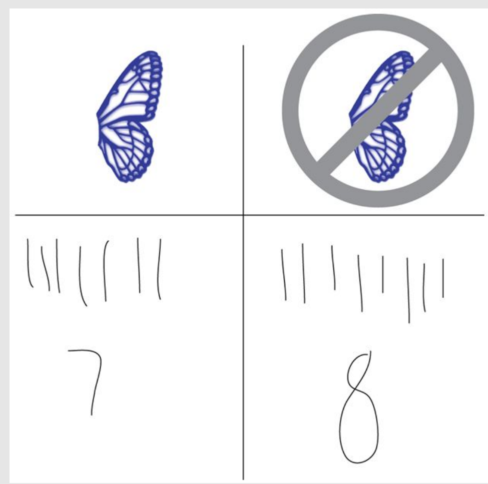
A single group of specimens might be classified in different ways, depending on which attribute has been identified as the attribute of interest. For example, some specimens might be insects, while others are not insects. Some specimens might live on land, while others live in water.

K.CC.5 Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

K.CC.6 Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.

K.CC.7 Compare two numbers between 1 and 10 presented as written numerals.

Sorting categorical data



The marks represent individual data points. The two category counts, 7 and 8, are a numerical summary *of the data.*

Grade 1

Students in Grade 1 begin to organize and represent categorical data. For example, if a collection of specimens is sorted into two piles based on which specimens have wings and which do not, students might represent the two piles of specimens on a piece of paper, by making a group of marks for each pile, as shown below (the marks could also be circles, for example). The groups of marks should be clearly labeled to reflect the attribute in question.

The work shown in the figure is the result of an intricate process. At first, we have before us a jumble of specimens with many attributes. Then there is a narrowing of attention to a single attribute (wings or not). Then the objects might be arranged into piles. The arranging of objects into piles is then mirrored in the arranging of marks into groups. In the end, each mark represents an object; its position in one column or the other indicates whether or not that object has a given attribute.

There is no single correct way to represent categorical data-and the Standards do not require Grade 1 students to use any specific format. However, students should be familiar with mark schemes like the one shown in the figure. Another format that might be useful in Grade 1 is a picture graph in which one picture represents one object. (Note that picture graphs are not an expectation in the Standards until Grade 2.) If different students devise different ways to represent the same data set, then the class might discuss relative strengths and weaknesses of each scheme (MP5).

Students' data work in Grade 1 has important connections to addition and subtraction, as noted in Table 1. Students in Grade 1 can ask and answer questions about categorical data based on a representation of the data. For example, with reference to the figure above, a student might ask how many specimens there were altogether, representing this problem by writing an equation such as $7 + 8 = \square$. Students can also ask and answer questions leading to other kinds of addition and subtraction problems (1.OA), such as compare problems or problems involving the addition of three numbers (for situations with three categories).

Grade 2

Students in Grade 2 draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. They solve simple put-together, take-apart, and compare problems using information presented in a bar graph.[2.MD.10, 2.OA.1]

The illustration shows an activity in which students make a bar graph to represent categorical data, then solve addition and subtraction problems based on the data. Students might use scissors to cut out the pictures of each organism and then sort the organisms into piles by category. Category counts might be recorded efficiently in the form of a table.

A bar graph representing categorical data displays no additional information beyond the category counts. In such a graph, the bars are a way to make the category counts easy to interpret visually. Thus, the word problem in part 4 could be solved without drawing a bar graph, just by using the category counts. The problem could even be cast entirely in words, without the accompanying picture: "There are 9 insects, 4 spiders, 13 vertebrates, and 2 organisms of other kinds. How many more spiders would there have to be in order for the number of spiders to equal the number of vertebrates?" Of course, in solving this problem, students would not need to participate in categorizing data or representing it.

Scales in bar graphs Consider the two bar graphs shown to the right, in which the bars are oriented vertically. (Bars in a bar graph can also be oriented horizontally, in which case the following discussion would be modified in the obvious way.) Both of these bar graphs represent the same data set.

These examples illustrate that the horizontal axis in a bar graph of categorical data is not a scale of any kind; position along the horizontal axis has no numerical meaning. Thus, the horizontal position and ordering of the bars are not determined by the data.•

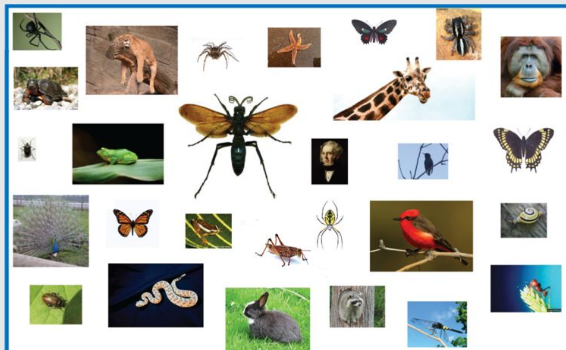
However, the vertical axes in these graphs do have numerical meaning. In fact, the vertical axes in these graphs are segments of number line diagrams. We might think of the vertical axis as a "count scale" (a scale showing counts in whole numbers)—as opposed to a measurement scale, which can be subdivided into fractions of a measurement unit.

Because the count scale in a bar graph is a segment of a number line diagram, when we answer a question such as "How many

2.MD.10 Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph.

2.OA.1 Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

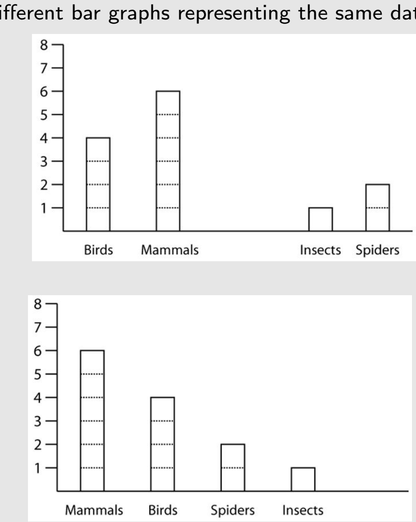
Activity for representing categorical data



1. How many organisms in the picture belong to each of the following categories: (a) insects (six legs); (b) spiders (eight legs); (c) vertebrates (backbone); (d) other.
2. To check your answer, do your counts add up to the correct total?
3. When you are sure your counts are correct, show them as a bar graph.
4. Alexa added more spiders to the picture until the number of spiders was the same as the number of vertebrates. How many spiders did she add?

Students might reflect on the way in which the category counts in part 1 of the activity enable them to efficiently solve the word problem in part 4. (The word problem in part 4 would be difficult to solve directly using just the array of images.)

Different bar graphs representing the same data set



• To minimize potential confusion, it might help to avoid presenting students with examples of categorical data in which the categories are named using numerals, e.g., "Candidate 1," "Candidate 2," "Candidate 3." This will ensure that the only numbers present in the display are found along the count scale.

more birds are there than spiders?" we are finding differences on a number line diagram.[2.MD.6]

When drawing bar graphs on grid paper, the tick marks on the count scale should be drawn at intersections of the gridlines. The tops of the bars should reach the respective gridlines of the appropriate tick marks. When drawing picture graphs on grid paper, the pictures representing the objects should be drawn in the squares of the grid paper.

Students could discuss ways in which bar orientation (horizontal or vertical), order, thickness, spacing, shading, colors, and so forth make the bar graphs easier or more difficult to interpret. By middle school, students could make thoughtful design choices about data displays, rather than just accepting the defaults in a software program (MP5).

Grade 3

In Grade 3, the most important development in data representation for categorical data is that students now draw picture graphs in which each picture represents more than one object, and they draw bar graphs in which the height of a given bar in tick marks must be multiplied by the scale factor in order to yield the number of objects in the given category. These developments connect with the emphasis on multiplication in this grade.

At the end of Grade 3, students can draw a scaled picture graph or a scaled bar graph to represent a data set with several categories (six or fewer categories).[3.MD.3] They can solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs.[3.OA.3,3.OA.8] See the examples in the margin, one of which involves categories of shapes.[3.G.1] As in Grade 2, category counts might be recorded efficiently in the form of a table.

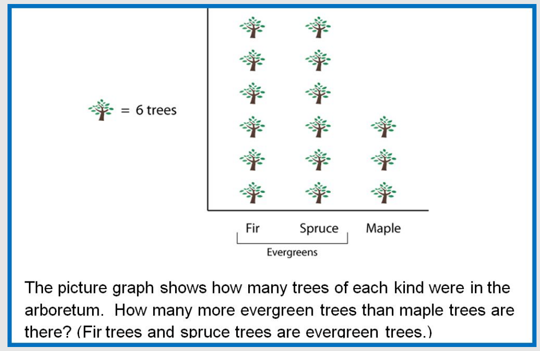
Students can gather categorical data in authentic contexts, including contexts arising in their study of science, history, health, and so on. Of course, students do not have to generate the data every time they work on making bar graphs and picture graphs. That would be too time-consuming. After some experiences in generating the data, most work in producing bar graphs and picture graphs can be done by providing students with data sets. The Standards in Grades 1–3 do not require students to gather categorical data.

Where the Categorical Data Progression is heading

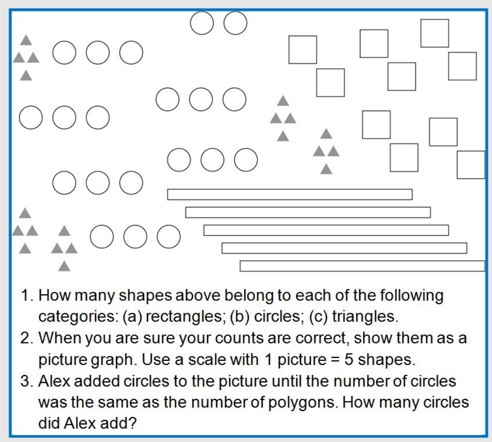
Students' work with categorical data in early grades will develop into later work with bivariate categorical data and two-way tables in eighth grade. "Bivariate categorical data" are data that are categorized according to two attributes. For example, if there is an outbreak of stomach illness on a cruise ship, then passengers might be sorted in two different ways: by determining who got sick and

[2.MD.6] Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

A problem about interpreting a scaled picture graph



Problems involving categorical data



[3.MD.3] Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs.

[3.OA.3] Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities,. . .

[3.OA.8] Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity . . .

[3.G.1] Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals . . .

who didn't, and by determining who ate the shellfish and who didn't. This double categorization—normally shown in the form of a two-way table—might show a strong positive or negative association, in which case it might used to support or contest (but not prove or disprove) a claim about whether the shellfish was the cause of the illness.[8.SP.4]

[8.SP.4]Understand that patterns of association can also be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table. Construct and interpret a two-way table summarizing data on two categorical variables collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible association between the two variables.

9

Measurement Data

Grade 2

Students in Grade 2 measure lengths to generate a set of measurement data. 2.MD.1 For example, each student might measure the length of his or her arm in centimeters, or every student might measure the height of a statue in inches. (Students might also generate their own ideas about what to measure.) The resulting data set will be a list of observations, for example as shown in the margin on the following page for the scenario of 28 students each measuring the height of a statue. (This is a larger data set than students would normally be expected to work with in elementary grades.)

2.MD.1 Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.

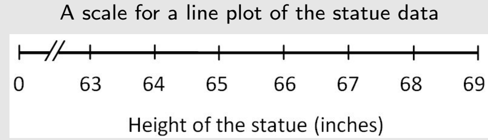
How might one summarize this data set or display it visually? Because students in Grade 2 are already familiar with categorical data and bar graphs, a student might find it natural to summarize this data set by viewing it in terms of categories—the categories in question being the six distinct height values which appear in the data (63 inches, 64 inches, 65 inches, 66 inches, 67 inches, and 69 inches). For example, the student might want to say that there are four observations in the "category" of 67 inches. However, it is important to recognize that 64 inches is not a category like "spiders." Unlike "spiders," 63 inches is a numerical value with a measurement unit. That difference is why the data in this table are called measurement data.

A display of measurement data must present the measured values with their appropriate magnitudes and spacing on the measurement scale in question (length, temperature, liquid capacity, etc.). One method for doing this is to make a *line plot*. This activity connects with other work students are doing in measurement in Grade 2: representing whole numbers on number line diagrams, and representing sums and differences on such diagrams.2.MD.5,2.MD.6

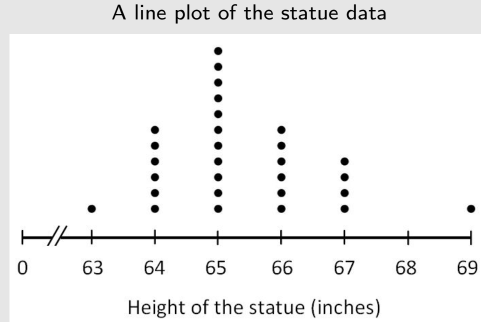
2.MD.5 Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

2.MD.6 Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

A scale for a line plot of the statue data



A line plot of the statue data



To make a line plot from the data in the table, the student can ascertain the greatest and least values in the data: 63 inches and 69 inches. The student can draw a segment of a number line diagram that includes these extremes, with tick marks indicating specific values on the measurement scale.

Note that the value 68 inches, which was not present in the data set, has been written in proper position midway between 67 inches and 69 inches. (This need to fill in gaps does not exist for a categorical data set; there no "gap" between categories such as fish and spiders!)

Having drawn the number line diagram, the student can proceed through the data set recording each observation by drawing a symbol, such as a dot, above the proper tick mark. If a particular data value appears many times in the data set, dots will "pile up" above that value. There is no need to sort the observations, or to do any counting of them, before producing the line plot. (In fact, one could even assemble the line plot as the data are being collected, at the expense of having a record of who made what measurement. Students might discuss whether such a record is valuable and why.)

Students might enjoy discussing and interpreting visual features of line plots, such as the "outlier" value of 69 inches in this line plot. (Did student #13 make a serious error in measuring the statue's height? Or in fact is student #13 the only person in the class who measured the height correctly?) However, in Grade 2 the only requirement of the Standards dealing with measurement data is that students generate measurement data and build line plots to display the resulting data sets. (Students do not have to generate the data every time they work on making line plots. That would be too time-consuming. After some experiences in generating the data, most work in producing line plots can be done by providing students with data sets.)

Grid paper might not be as useful for drawing line plots as it is for bar graphs, because the count scale on a line plot is seldom shown for the small data sets encountered in the elementary grades. Additionally, grid paper is usually based on a square grid, but the count scale and the measurement scale of a line plot are conceptually distinct, and there is no need for the measurement unit on the measurement scale to be drawn the same size as the counting unit on the count scale.

Grade 3

In Grade 3, students are beginning to learn fraction concepts (3.NF). They understand fraction equivalence in simple cases, and they use visual fraction models to represent and order fractions. Grade 3 students also measure lengths using rulers marked with halves and fourths of an inch. They use their developing knowledge of fractions and number lines to extend their work from the previous grade by working with measurement data involving fractional measurement values.

For example, every student in the class might measure the height of a bamboo shoot growing in the classroom, leading to the data set shown in the table. (Again, this illustration shows a larger data set than students would normally work with in elementary grades.)

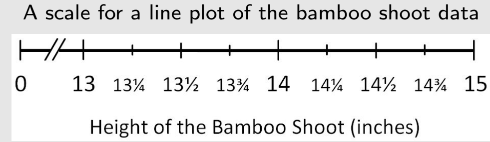
To make a line plot from the data in the table, the student can ascertain the greatest and least values in the data: $13\frac{1}{2}$ inches and $14\frac{3}{4}$ inches. The student can draw a segment of a number line diagram that includes these extremes, with tick marks indicating specific values on the measurement scale. This is just like part of the scale on a ruler.

Having drawn the number line diagram, the student can proceed through the data set recording each observation by drawing a symbol, such as a dot, above the proper tick mark. As with Grade 2 line plots, if a particular data value appears many times in the data set, dots will "pile up" above that value. There is no need to sort the

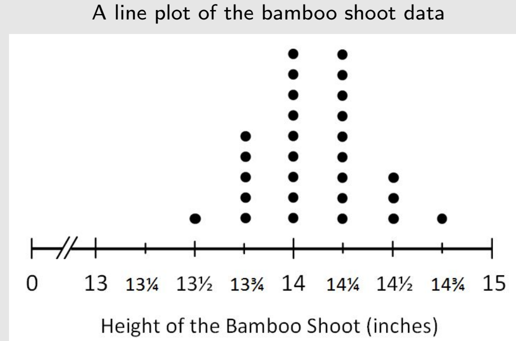
Students' measurements of a statue and of a bamboo shoot

Statue measurements		Bamboo shoot measurements	
Student's initials	Student's measured value (inches)	Student's initials	Height value (inches)
W.B.	64	W.B.	13 3/4
D.W.	65	D.W.	14 1/2
H.D.	65	H.D.	14 1/4
G.W.	65	G.W.	14 3/4
V.Y.	67	V.Y.	14 1/4
T.T.	66	T.T.	14 1/2
D.F.	67	D.F.	14
B.H.	65	B.H.	13 1/2
H.H.	63	H.H.	14 1/4
V.H.	64	V.H.	14 1/4
I.O.	64	I.O.	14 1/4
W.N.	65	W.N.	14
B.P.	69	B.P.	14 1/2
V.A.	65	V.A.	13 3/4
H.L.	66	H.L.	14
O.M.	64	O.M.	13 3/4
L.E.	65	L.E.	14 1/4
M.J.	66	M.J.	13 3/4
T.D.	66	T.D.	14 1/4
K.P.	64	K.P.	14
H.N.	65	H.N.	14
W.M.	67	W.M.	14
C.Z.	64	C.Z.	13 3/4
J.I.	66	J.I.	14
M.S.	66	M.S.	14 1/4
T.C.	65	T.C.	14
G.V.	67	G.V.	14
O.F.	65	O.F.	14 1/4

A scale for a line plot of the bamboo shoot data



A line plot of the bamboo shoot data



observations, or to do any counting of them, before producing the line plot.

Students can pose questions about data presented in line plots, such as how many students obtained measurements larger than $14\frac{1}{4}$ inches.

Grades 4 and 5

Grade 4 students learn elements of fraction equivalence[4.NF.1] and arithmetic, including multiplying a fraction by a whole number[4.NF.4] and adding and subtracting fractions with like denominators.[4.NF.3] Students can use these skills to solve problems, including problems that arise from analyzing line plots. For example, with reference to the line plot above, students might find the difference between the greatest and least values in the data. (In solving such problems, students may need to label the measurement scale in eighths so as to produce like denominators. Decimal data can also be used in this grade.)

Grade 5 students grow in their skill and understanding of fraction arithmetic, including multiplying a fraction by a fraction,[5.NF.4] dividing a unit fraction by a whole number or a whole number by a unit fraction,[4.NF.7] and adding and subtracting fractions with unlike denominators.[5.NF.1] Students can use these skills to solve problems,[5.NF.2,5.NF.6,5.NF.7c] including problems that arise from analyzing line plots. For example, given five graduated cylinders with different measures of liquid in each, students might find the amount of liquid each cylinder would contain if the total amount in all the cylinders were redistributed equally. (Students in Grade 6 will view the answer to this question as the mean value for the data set in question.)

As in earlier grades, students should work with data in science and other subjects. Grade 5 students working in these contexts should be able to give deeper interpretations of data than in earlier grades, such as interpretations that involve informal recognition of pronounced differences in populations. This prefigures the work they will do in middle school involving distributions, comparisons of populations, and inference.

4.NF.1 Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

4.NF.4 Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

4.NF.3 Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

5.NF.4 Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

4.NF.7 Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual model.

5.NF.1 Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators.

5.NF.2 Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers.

5.NF.6 Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

5.NF.7c Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.

- c Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem.

Where the Measurement Data Progression is heading

Connection to Statistics and Probability By the end of Grade 5, students should be comfortable making line plots for measurement data and analyzing data shown in the form of a line plot. In Grade 6, students will take an important step toward statistical reasoning per se when they approach line plots as pictures of distributions with features such as clustering and outliers.

Students' work with line plots during the elementary grades develops in two distinct ways during middle school. The first development comes in sixth grade,[6.SP.4] when *histograms* are used.[1] Like

6.SP.4 Display numerical data in plots on a number line, including dot plots, histograms, and box plots.

line plots, histograms have a measurement scale and a count scale; thus, a histogram is a natural evolution of a line plot and is used for similar kinds of data (univariate measurement data, the kind of data discussed above).

The other evolution of line plots in middle school is arguably more important. It involves the graphing of bivariate measurement data.[8.SP.1-3] "Bivariate measurement data" are data that represent two measurements. For example, if you take a temperature reading every ten minutes, then every data point is a measurement of temperature as well as a measurement of time. Representing two measurements requires two measurement scales—or in other words, a coordinate plane in which the two axes are each marked in the relevant measurement units. Representations of bivariate measurement data in the coordinate plane are called *scatter plots*. In the case where one axis is a time scale, they are called *time graphs* or *line graphs*. Time graphs can be used to visualize trends over time, and scatter plots can be used to discover associations between measured variables in general.

8.SP.1 Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative association, linear association, and nonlinear association.

8.SP.2 Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line.

8.SP.3 Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept.

Connection to the Number System The Standards do not explicitly require students to create time graphs. However, it might be considered valuable to expose students to time series data and to time graphs as part of their work in meeting standard 6.NS.8. For example, students could create time graphs of temperature measured each hour over a 24-hour period, where, to ensure a strong connection to rational numbers, temperature values might cross from positive to negative during the night and back to positive the next day. It is traditional to connect ordered pairs with line segments in such a graph, not in order to make any claims about the actual temperature value at unmeasured times, but simply to aid the eye in perceiving trends.

6.NS.8 Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

[1] To display a set of measurement data with a histogram, specify a set of non-overlapping intervals along the measurement scale. Then, instead of showing each individual measurement as a dot, use a bar oriented along the count scale to indicate the number of measurements lying within each interval on the measurement scale. A histogram is thus a little like a bar graph for categorical data, except that the "categories" are successive intervals along a measurement scale. (Note that the Standards follow the GAISE report in reserving the term "categorical data" for non-numerical categories. In the Standards, as in GAISE (see p. 35), bar graphs are for categorical data with non-numerical categories, while histograms are for measurement data which has been grouped by intervals along the measurement scale.)

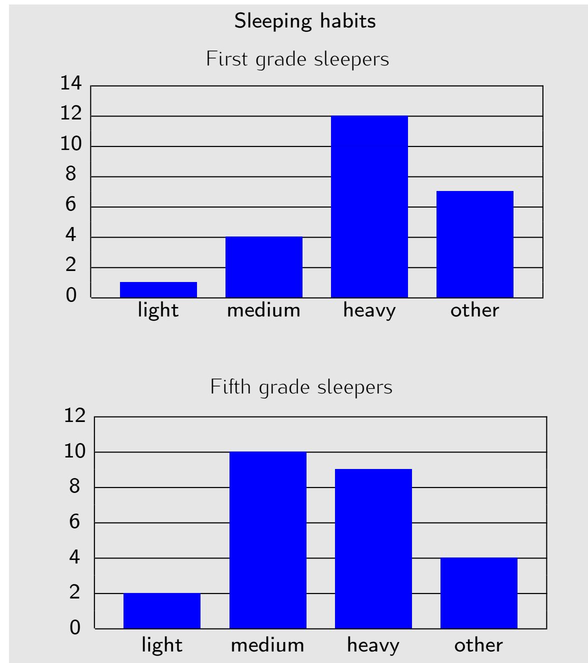
Appendix: Additional Examples

These examples show some rich possibilities for data work in K–8. The examples are not shown by grade level because each includes some aspects that go beyond the expectations stated in the Standards.

Example 1. Comparing bar graphs

Are younger students lighter sleepers than older students? To study this question a class first agreed on definitions for light, medium and heavy sleepers and then collected data from first and fifth grade students on their sleeping habits. The results are shown in the margin.

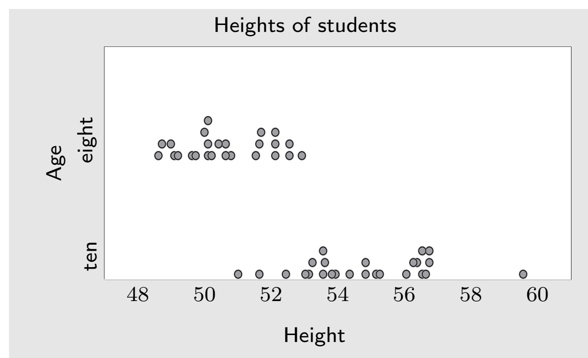
How do the patterns differ? What is the typical value for first graders? What is the typical value for fifth graders? Which of these groups appears to be the heavier sleepers?



Example 2. Comparing line plots

Fourth grade students interested in seeing how heights of students change for kids around their age measured the heights of a sample of eight-year-olds and a sample of ten-year-olds. Their data are plotted in the margin.

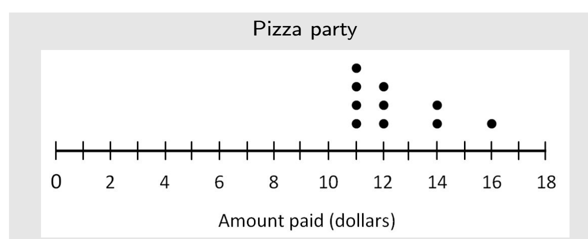
Describe the key differences between the heights of these two age groups. What would you choose as the typical height of an eight-year-old? A ten-year-old? What would you say is the typical number of inches of growth from age eight to age ten?



Example 3. Fair share averaging

Ten students decide to have a pizza party and each is asked to bring his or her favorite pizza. The amount paid (in dollars) for each pizza is shown in the plot to the right.

Each of the ten is asked to contribute an equal amount (his or her fair share) to the cost of the pizza. Where does that fair share amount lie on the plot? Is it closer to the smaller values or the large one? Now, two more students show up for the party and they have contributed no pizza. Plot their values on the graph and calculate a new fair share. Where does it lie on the plot? How many more students without pizza would have to show up to bring the fair share cost below $8.00?



Progressions for the Common Core State Standards in Mathematics (draft)

23 June 2012

K–5, Geometric Measurement

1

Overview

Geometric measurement connects the two most critical domains of early mathematics, geometry and number, with each providing conceptual support to the other. Measurement is central to mathematics, to other areas of mathematics (e.g., laying a sensory and conceptual foundation for arithmetic with fractions), to other subject matter domains, especially science, and to activities in everyday life. For these reasons, measurement is a core component of the mathematics curriculum.

Measurement is the process of assigning a number to a magnitude of some attribute shared by some class of objects, such as length, relative to a unit. Length is a *continuous* attribute—a length can always be subdivided in smaller lengths. In contrast, we can count 4 apples exactly—cardinality is a discrete attribute. We can add the 4 apples to 5 other apples and know that the result is exactly 9 apples. However, the *weight* of those apples is a continuous attribute, and scientific measurement with tools gives only an approximate measurement—to the nearest pound (or, better, kilogram) or the nearest $1/100^{\text{th}}$ of a pound, but always with some error.•

• The Standards do not differentiate between weight and mass. Technically, mass is the amount of matter in an object. Weight is the force exerted on the body by gravity. On the earth's surface, the distinction is not important (on the moon, an object would have the same mass, would weight less due to the lower gravity).

Before learning to measure attributes, children need to recognize them, distinguishing them from other attributes. That is, the attribute to be measured has to "stand out" for the student and be discriminated from the undifferentiated sense of amount that young children often have, labeling greater lengths, areas, volumes, and so forth, as "big" or "bigger."

Students then can become increasingly competent at *direct comparison*—comparing the amount of an attribute in two objects without measurement. For example, two students may stand back to back to directly compare their heights. In many circumstances, such direct comparison is impossible or unwieldy. Sometimes, a third object can be used as an intermediary, allowing *indirect comparison*. For example, if we know that Aleisha is taller than Barbara and that

[1] This progression concerns Measurement and Data standards related to geometric measurement. The remaining Measurement and Data standards are discussed in the K–3 Categorical Data and Grades 2–5 Measurement Data Progressions.

Barbara is taller than Callie, then we know (due to the transitivity of "taller than") that Aleisha is taller than Callie, even if Aleisha and Callie never stand back to back.•

The purpose of measurement is to allow indirect comparisons of objects' amount of an attribute using numbers. An attribute of an object is measured (i.e., assigned a number) by comparing it to an amount of that attribute held by another object. One measures length with length, mass with mass, torque with torque, and so on. In geometric measurement, a unit is chosen and the object is subdivided or partitioned by copies of that unit and, to the necessary degree of precision, units subordinate to the chosen unit, to determine the number of units and subordinate units in the partition.

Personal benchmarks, such as "tall as a doorway" build students' intuitions for amounts of a quantity and help them use measurements to solve practical problems. A combination of internalized units and measurement processes allows students to develop increasing accurate estimation competencies.

Both in measurement and in estimation, the concept of *unit* is crucial. The concept of basic (as opposed to subordinate) unit just discussed is one aspect of this concept. The basic unit can be informal (e.g., about a car length) or standard (e.g., a meter). The distinction and relationship between the notion of discrete "1" (e.g., one apple) and the continuous "1" (e.g., one inch) is important mathematically and is important in understanding number line diagrams (e.g., see Grade 2) and fractions (e.g., see Grade 3). However, there are also superordinate units or "units of units." A simple example is a kilometer consisting of 1,000 meters. Of course, this parallels the number concepts students must learn, as understanding that tens and hundreds are, respectively, "units of units" and "units of units of units" (i.e., students should learn that 100 can be simultaneously considered as 1 hundred, 10 tens, and 100 ones).

Students' understanding of an attribute that is measured with derived units is dependent upon their understanding that attribute as entailing other attributes simultaneously. For example,

- Area as entailing two lengths, simultaneously;
- Volume as entailing area and length (and thereby three lengths), simultaneously.

Scientists measure many types of attributes, from hardness of minerals to speed. This progression emphasizes the geometric attributes of length, area, and volume. Nongeometric attributes such as weight, mass, capacity, time, and color, are often taught effectively in science and social studies curricula and thus are not extensively discussed here. Attributes derived from two different attributes, such as speed (derived from distance and time), are discussed in the high school Number and Quantity Progression and in the 6-7 Ratio and Proportion Progression.

• "Transitivity" abbreviates the Transitivity Principle for Indirect Measurement stated in the Standards as:

> If the length of object A is greater than the length of object B, and the length of object B is greater than the length of object C, then the length of object A is greater than the length of object C. This principle applies to measurement of other quantities as well.

Students should apply the principle of transitivity of measurement to make indirect comparisons, but they need not use this technical term.

Length is a characteristic of an object found by quantifying how far it is between the endpoints of the object. "Distance" is often used similarly to quantify how far it is between any two points in space. Measuring length or distance consists of two aspects, choosing a unit of measure and *subdividing* (mentally and physically) the object by that unit, placing that unit end to end (*iterating*) alongside the object. The length of the object is the number of units required to iterate from one end of the object to the other, without gaps or overlaps.

Length is a core concept for several reasons. It is the basic geometric measurement. It is also involved in area and volume measurement, especially once formulas are used. Length and unit iteration are critical in understanding and using the number line in Grade 3 and beyond (see the Number and Operations—Fractions Progression). Length is also one of the most prevalent metaphors for quantity and number, e.g., as the master metaphor for magnitude (e.g., vectors, see the Number and Quantity Progression). Thus, length plays a special role in this progression.

Area is an amount of two-dimensional surface that is contained within a plane figure. Area measurement assumes that congruent figures enclose equal areas, and that area is *additive*, i.e., the area of the union of two regions that overlap only at their boundaries is the sum of their areas. Area is measured by tiling a region with a two-dimensional unit (such as a square) and parts of the unit, without gaps or overlaps. Understanding how to spatially structure a two-dimensional region is an important aspect of the progression in learning about area.

Volume is an amount of three-dimensional space that is contained within a three-dimensional shape. Volume measurement assumes that congruent shapes enclose equal volumes, and that volume is *additive*, i.e., the volume of the union of two regions that overlap only at their boundaries is the sum of their volumes. Volume is measured by packing (or tiling, or tessellating) a region with a three-dimensional unit (such as a cube) and parts of the unit, without gaps or overlaps. Volume not only introduces a third dimension and thus an even more challenging spatial structuring, but also complexity in the nature of the materials measured. That is, solid units might be "packed," such as cubes in a three-dimensional array or cubic meters of coal, whereas liquids "fill" three-dimensional regions, taking the shape of a container, and are often measured in units such as liters or quarts.

A final, distinct, geometric attribute is *angle measure*. The size of an angle is the amount of rotation between the two rays that form the angle, sometimes called the sides of the angles.

Finally, although the attributes that we measure differ as just described, it is important to note: *central characteristics of measurement are the same for all of these attributes.* As one more testament to these similarities, consider the following side-by-side comparison of the Standards for measurement of area in Grade 3 and the measurement of volume in Grade 5.

Grade 3

Understand concepts of area and relate area to multiplication and to addition.

3.MD.5. Recognize area as an attribute of plane figures and understand concepts of area measurement.

a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.
b. A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.

3.MD.6. Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

3.MD.7. Relate area to the operations of multiplication and addition.

a. Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.
b. Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.
c. Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.
d. Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.

Grade 5

Understand concepts of volume and relate volume to multiplication and to addition.

5.MD.3 Recognize volume as an attribute of solid figures and understand concepts of volume measurement.

a. A cube with side length 1 unit, called a "unit cube," is said to have "one cubic unit" of volume, and can be used to measure volume.
b. A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.

5.MD.4 Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.

5.MD.5 Relate volume to the operations of multiplication and addition and solve real world and mathematical problems involving volume.

a. Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.
b. Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real world and mathematical problems.
c. Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.

Kindergarten

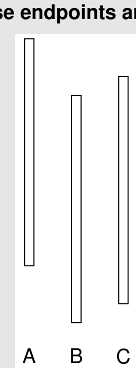
Describe and compare measurable attributes Students often initially hold undifferentiated views of measurable attributes, saying that one object is "bigger" than another whether it is longer, or greater in area, or greater in volume, and so forth. For example, two students might both claim their block building is "the biggest." Conversations about how they are comparing—one building may be taller (greater in length) and another may have a larger base (greater in area)—help students learn to discriminate and name these measureable attributes. As they discuss these situations and compare objects using different attributes, they learn to distinguish, label, and describe several measureable attributes of a single object.[K.MD.1] Thus, teachers listen for and extend conversations about things that are "big," or "small," as well as "long," "tall," or "high," and name, discuss, and demonstrate with gestures the attribute being discussed (length as extension in one dimension is most common, but area, volume, or even weight in others).

K.MD.1 Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.

Length Of course, such conversations often occur in comparison situations ("He has more than me!"). Kindergartners easily directly compare lengths in simple situations, such as comparing people's heights, because standing next to each other automatically aligns one endpoint.[K.MD.2] However, in other situations they may initially compare only one endpoint of objects to say which is longer. Discussing such situations (e.g., when a child claims that he is "tallest" because he is standing on a chair) can help students resolve and coordinate perceptual and conceptual information when it conflicts. Teachers can reinforce these understandings, for example, by holding two pencils in their hand showing only one end of each, with the longer pencil protruding less. After asking if they can tell which pencil is longer, they reveal the pencils and discuss whether children were "fooled." The necessity of aligning endpoints can be explicitly addressed and then re-introduced in the many situations throughout the day that call for such comparisons. Students can also make such comparisons by moving shapes together to see which has a longer side.

K.MD.2 Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference.

Sticks whose endpoints are not aligned



When shown this figure and asked which is "the longest stick," students may point to A because it "sticks out the farthest." Similarly, they may recognize a 12-inch vertical line as "tall" and a 12-inch horizontal line as "long" but not recognize that the two are the same length.

Even when students seem to understand length in such activities, they may not conserve length. That is, they may believe that if one of two sticks of equal lengths is vertical, it is then longer than the other, horizontal, stick. Or, they may believe that a string, when bent or curved, is now shorter (due to its endpoints being closer to each other). Both informal and structured experiences, including demonstrations and discussions, can clarify how length is maintained, or conserved, in such situations. For example, teachers and students might rotate shapes to see its sides in different orientations. As with number, learning and using language such as "It looks longer, but it really isn't longer" is helpful.

Students who have these competencies can engage in experiences that lay the groundwork for later learning. Many can begin to learn to compare the lengths of two objects using a third object, order lengths, and connect number to length. For example, informal experiences such as making a road "10 blocks long" help students build a foundation for measuring length in the elementary grades. See the Grade 1 section on length for information about these important developments.

Area and volume Although area and volume experiences are not instructional foci for Kindergarten, they are attended to, at least to distinguish these attributes from length, as previously described. Further, certain common activities can help build students' experiential foundations for measurement in later grades. Understanding area requires understanding this attribute as the amount of two-dimensional space that is contained within a boundary. Kindergartners might informally notice and compare areas associated with everyday activities, such as laying two pieces of paper on top of each other to find out which will allow a "bigger drawing." Spatial structuring activities described in the Geometry Progression, in which designs are made with squares covering rectilinear shapes also help to create a foundation for understanding area.

Similarly, kindergartners might compare the capacities of containers informally by pouring (water, sand, etc.) from one to the other. They can try to find out which holds the most, recording that, for example, the container labeled "J" holds more than the container labeled "D" because when J was poured into D it overflowed. Finally, in play, kindergartners might make buildings that have layers of rectangular arrays. Teachers aware of the connections of such activities to later mathematics can support students' growth in multiple domains (e.g., development of self-regulation, social-emotional, spatial, and mathematics competencies) simultaneously, with each domain supporting the other.

Grade 1

Length comparisons First graders should continue to use direct comparison—carefully, considering all endpoints—when that is appropriate. In situations where direct comparison is not possible or convenient, they should be able to use indirect comparison and explanations that draw on transitivity (MP3). Once they can compare lengths of objects by direct comparison, they could compare several items to a single item, such as finding all the objects in the classroom the same length as (or longer than, or shorter than) their forearm.[1.MD.1] Ideas of transitivity can then be discussed as they use a string to represent their forear's length. As another example, students can figure out that one path from the teachers' desk to the door is longer than another because the first path is longer than a length of string laid along the path, but the other path is shorter than that string. Transitivity can then be explicitly discussed: If *A* is longer than *B* and *B* is longer than *C*, then *A* must be longer than *C* as well.

1.MD.1 Order three objects by length; compare the lengths of two objects indirectly by using a third object.

Seriation Another important set of skills and understandings is ordering a set of objects by length.[1.MD.1] Such sequencing requires multiple comparisons. Initially, students find it difficult to seriate a large set of objects (e.g., more than 6 objects) that differ only slightly in length. They tend to order groups of two or three objects, but they cannot correctly combine these groups while putting the objects in order. Completing this task efficiently requires a systematic strategy, such as moving each new object "down the line" to see where it fits. Students need to understand that each object in a seriation is larger than those that come before it, and shorter than those that come after. Again, reasoning that draws on transitivity is relevant.

1.MD.1 Order three objects by length; compare the lengths of two objects indirectly by using a third object.

Such seriation and other processes associated with the measurement and data standards are important in themselves, but also play a fundamental role in students' development. The general reasoning processes of seriation, conservation (of length and number), and classification (which lies at the heart of the standards discussed in the K–3 Categorical Data Progression) predict success in early childhood as well as later schooling.

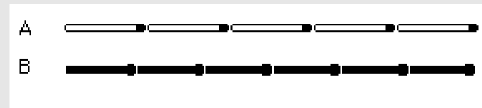
Measure lengths indirectly and by iterating length units Directly comparing objects, indirectly comparing objects, and ordering objects by length are important practically and mathematically, but they are not length measurement, which involves assigning a number to a length. Students learn to lay physical units such as centimeter or inch manipulatives end-to-end and count them to measure a length.[1.MD.2] Such a procedure may seem to adults to be straightforward, however, students may initially iterate a unit leaving gaps between subsequent units or overlapping adjacent units. For such students, measuring may be an activity of placing units along a

1.MD.2 Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps.

path in some manner, rather than the activity of covering a region or length with no gaps.

Also, students, especially if they lack explicit experience with continuous attributes, may make their initial measurement judgments based upon experiences counting discrete objects. For example, researchers showed children two rows of matches. The matches in each row were of different lengths, but there was a different number of matches in each so that the rows were the same length. Although, from the adult perspective, the lengths of the rows were the same, many children argued that the row with 6 matches was longer because it had more matches. They counted units (matches), assigning a number to a *discrete* attribute (cardinality). In measuring *continuous* attributes, the sizes of the units (white and dark matches) must be considered. First grade students can learn that objects used as basic units of measurement (e.g., "match-length") must be the same size.

Matches of different lengths



Row A is 5 matches long–when the unit of measurement is white matches. Row B is 6 matches long–when the unit of measurement is dark matches. From Inhelder, Sinclair, and Bovet, 1974, *Learning and the Development of Cognition*, *Harvard University Press.*

As with transitive reasoning tasks, using comparison tasks and asking children to compare results can help reveal the limitations of such procedures and promote more accurate measuring. However, students also need to see agreements. For example, understanding that the results of measurement and direct comparison have the same results encourages children to use measurement strategies.

Another important issue concerns the use of standard or nonstandard units of length. Many curricula or other instructional guides advise a sequence of instruction in which students compare lengths, measure with nonstandard units (e.g., paper clips), incorporate the use of manipulative standard units (e.g., inch cubes), and measure with a ruler. This approach is probably intended to help students see the need for standardization. However, the use of a variety of different length units, *before students understand the concepts, procedures, and usefulness of measurement*, may actually deter students' development. Instead, students might learn to measure correctly with standard units, and even learn to use rulers, before they can successfully use nonstandard units and understand relationships between different units of measurement. To realize that arbitrary (and especially mixed-size) units result in the same length being described by different numbers, a student must reconcile the varying lengths and numbers of arbitrary units. Emphasizing nonstandard units too early may defeat the purpose it is intended to achieve. Early use of many nonstandard units may actually interfere with students' development of basic measurement concepts required to understand the need for standard units. In contrast, using manipulative standard units, or even standard rulers, is less demanding and appears to be a more interesting and meaningful real-world activity for young students.

Thus, an instructional progression based on this finding would start by ensuring that students can perform direct comparisons. Then, children should engage in experiences that allow them to connect number to length, using manipulative units that have a standard unit of length, such as centimeter cubes. These can be labeled "length-units" with the students. Students learn to lay such physical units end-to-end and count them to measure a length. They compare the results of measuring to direct and indirect comparisons.

As they measure with these manipulative units, students discuss the concepts and skills involved (e.g., as previously discussed, not leaving space between successive length-units). As another example, students initially may not extend the unit past the endpoint of the object they are measuring. If students make procedural errors such as these, they can be asked to tell in a precise and elaborate manner *what* the problem is, *why* it leads to incorrect measurements, and *how* to fix it and measure accurately.

Measurement activities can also develop other areas of mathematics, including reasoning and logic. In one class, first graders were studying mathematics mainly through measurement, rather than counting discrete objects. They described and represented relationships among and between lengths (MP2, MP3), such as comparing two sticks and symbolizing the lengths as "$A < B$." This enabled them to reason about relationships. For example, after seeing the following statements recorded on the board, if $V > M$, then $M \neq V$, $V \neq M$, and $M < V$, one first-grader noted, "If it's an inequality, then you can write four statements. If it's equal, you can only write two"(MP8)

This indicates that with high-quality experiences (such as those described in the Grade 2 section on length), many first graders can also learn to use reasoning, connecting this to direct comparison, and to measurement performed by laying physical units end-to-end.

Area and volume: Foundations As in Kindergarten, area and volume are not instructional foci for first grade, but some everyday activities can form an experiential foundation for later instruction in these topics. For example, in later grades, understanding area requires seeing how to decompose shapes into parts and how to move and recombine the parts to make simpler shapes whose areas are already known (MP7). First graders learn the foundations of such procedures both in composing and decomposing shapes, discussed in the Geometry Progression, and in comparing areas in specific contexts. For example, paper-folding activities lend themselves not just to explorations of symmetry but also to equal-area congruent parts. Some students can compare the area of two pieces of paper by cutting and overlaying them. Such experiences provide only initial development of area concepts, but these key foundations are important for later learning.

Volume can involve liquids or solids. This leads to two ways to measure volume, illustrated by "packing" a space such as a three-dimensional array with cubic units and "filling" with iterations of a fluid unit that takes the shape of the container (called liquid volume). Many first graders initially perceive filling as having a one-dimensional unit structure. For example, students may simply "read off" the measure on a graduated cylinder. Thus, in a science or "free time" activity, students might compare the volume of two containers in at least two ways. They might pour each into a graduated cylinder to compare the measures. Or they might practice indirect comparison using transitive reasoning by using a third container to compare the volumes of the two containers. By packing unit cubes into containers into which cubes fit readily, students also can lay a foundation for later "packing" volume.

Grade 2

Measure and estimate lengths in standard units Second graders learn to measure length with a variety of tools, such as rulers, meter sticks, and measuring tapes.[2.MD.1] Although this appears to some adults to be relatively simple, there are many conceptual and procedural issues to address. For example, students may begin counting at the numeral "1" on a ruler. The numerals on a ruler may signify to students when to start counting, rather than the amount of space that has already been covered. It is vital that students learn that "one" represents the space from the beginning of the ruler to the hash mark, not the hash mark itself. Again, students may not understand that units must be of equal size. They will even measure with tools subdivided into units of different sizes and conclude that quantities with more units are larger.

To learn measurement concepts and skills, students might use both simple rulers (e.g., having only whole units such as centimeters or inches) and physical units (e.g., manipulatives that are centimeter or inch lengths). As described for Grade 1, teachers and students can call these "length-units." Initially, students lay multiple copies of the same physical unit end-to-end along the ruler. They can also progress to iterating with one physical unit (i.e., repeatedly marking off its endpoint, then moving it to the next position), even though this is more difficult physically and conceptually. To help them make the transition to this more sophisticated understanding of measurement, students might draw length unit marks along sides of geometric shapes or other lengths to see the unit lengths. As they measure with these tools, students with the help of the teacher discuss the concepts and skills involved, such as the following.

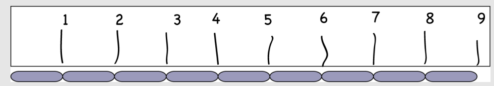
- *length-unit iteration*. E.g., not leaving space between successive length-units;
- *accumulation of distance*. Understanding that the counting "eight" when placing the last length-unit means the space covered by 8 length-units, rather then just the eighth length-unit (note the connection to cardinality[K.CC.4]);
- *alignment of zero-point*. Correct alignment of the zero-point on a ruler as the beginning of the total length, including the case in which the 0 of the ruler is not at the edge of the physical ruler;
- *meaning of numerals on the ruler*. The numerals indicate the number of length units so far;
- *connecting measurement with physical units and with a ruler*. Measuring by laying physical units end-to-end or iterating a physical unit and measuring with a ruler both focus on finding the total number of unit lengths.

2.MD.1 Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.

K.CC.4 Understand the relationship between numbers and quantities; connect counting to cardinality.

Students also can learn accurate procedures and concepts by drawing simple unit rulers. Using copies of a single length-unit such as inch-long manipulatives, they mark off length-units on strips of paper, explicitly connecting measurement with the ruler to measurement by iterating physical units. Thus, students' first rulers should be simply ways to help count the iteration of length-units. Frequently comparing results of measuring the same object with manipulative standard units and with these rulers helps students connect their experiences and ideas. As they build and use these tools, they develop the ideas of length-unit iteration, correct alignment (with a ruler), and the zero-point concept (the idea that the zero of the ruler indicates one endpoint of a length). These are reinforced as children compare the results of measuring to compare to objects with the results of directly comparing these objects.

Using a unit to draw a ruler



Students use a standard unit (shown in below the ruler) to make rulers, helping them to understand the meaning of the markings on rulers.

Similarly, discussions might frequently focus on "*What* are you counting?" with the answer being "length-units" or "centimeters" or the like. This is especially important because counting discrete items often convinces students that the size of things counted does not matter (there could be exactly 10 toys, even if they are different sizes). In contrast, for measurement, unit size is critical, so teachers are advised to plan experiences and reflections on the use of other units and length-units in various discrete counting and measurement contexts. Given that counting discrete items often correctly teaches students that the length-unit size does not matter, so teachers are advised to plan experiences and reflections on the use of units in various discrete counting and measurement contexts. For example, a teacher might challenge students to consider a fictitious student's measurement in which he lined up three large and four small blocks and claimed a path was "seven blocks long." Students can discuss whether he is correct or not.

Second graders also learn the concept of the inverse relationship between the size of the unit of length and the number of units required to cover a specific length or distance.[2.MD.2] For example, it will take more centimeter lengths to cover a certain distance than inch lengths because inches are the larger unit. Initially, students may not appreciate the need for identical units. Previously described work with manipulative units of standard measure (e.g., 1 inch or 1 cm), along with related use of rulers and consistent discussion, will help children learn both the concepts and procedures of linear measurement. Thus, second grade students can learn that the larger the unit, the fewer number of units in a given measurement (as was illustrated on p. 9). That is, for measurements of a given length there is an inverse relationship between the size of the unit of measure and the number of those units. This is the time that measuring and reflecting on measuring the same object with different units, both standard and nonstandard, is likely to be most productive (see the discussion of this issue in the Grade 1 section on length). Results of measuring with different nonstandard length-units can be explicitly compared. Students also can use the concept of unit to make

[2.MD.2] Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.

inferences about the relative sizes of objects; for example, if object A is 10 regular paperclips long and object B is 10 jumbo paperclips long, the number of units is the same, but the units have different sizes, so the lengths of A and B are different.

Second graders also learn to combine and compare lengths using arithmetic operations. That is, they can add two lengths to obtain the length of the whole and subtract one length from another to find out the difference in lengths.[2.MD.4] For example, they can use a simple unit ruler or put a length of connecting cubes together to measure first one modeling clay "snake," then another, to find the total of their lengths. The snakes can be laid along a line, allowing students to compare the measurement of that length with the sum of the two measurements. Second graders also begin to apply the concept of length in less obvious cases, such as the width of a circle, the length and width of a rectangle, the diagonal of a quadrilateral, or the height of a pyramid. As an arithmetic example, students might measure all the sides of a table with unmarked (foot) rulers to measure how much ribbon they would need to decorate the perimeter of the table.[2.MD.5] They learn to measure two objects and subtract the smaller measurement from the larger to find how much longer one object is than the other.

Second graders can also learn to represent and solve numerical problems about length using tape or number-bond diagrams. (See p. 16 of the Operations and Algebraic Thinking Progression for discussion of when and how these diagrams are used in Grade 1.) Students might solve two-step numerical problems at different levels of sophistication (see p. 18 of the Operations and Algebraic Thinking Progression for similar two-step problems involving discrete objects). Conversely, "missing measurements" problems about length may be presented with diagrams.

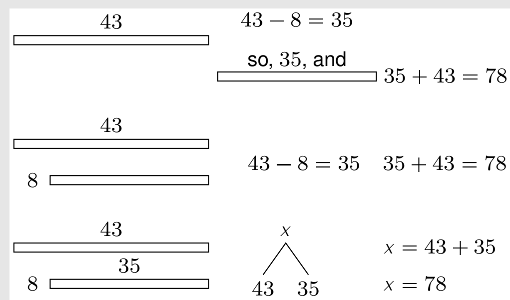
These understandings are essential in supporting work with number line diagrams.[2.MD.6] That is, to use a number line diagram to understand number and number operations, students need to understand that number line diagrams have specific conventions: the use of a single position to represent a whole number and the use of marks to indicate those positions. They need to understand that a number line diagram is like a ruler in that consecutive whole numbers are 1 unit apart, thus they need to consider the distances between positions and segments when identifying missing numbers. These understandings underlie students' successful use of number line diagrams. Students think of a number line diagram as a measurement model and use strategies relating to distance, proximity of numbers, and reference points.

After experience with measuring, second graders learn to estimate lengths.[2.MD.3] Real-world applications of length often involve estimation. Skilled estimators move fluently back and forth between written or verbal length measurements and representations of their corresponding magnitudes on a *mental ruler* (also called the "mental number line"). Although having real-world "benchmarks" is useful

[2.MD.4] Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

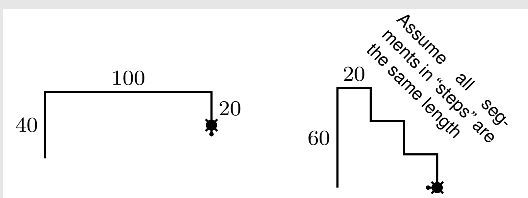
[2.MD.5] Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

Missing measurements problems



Different solution methods for "A girl had a 43 cm section of a necklace and another section that was 8 cm shorter than the first. How long the necklace would be if she combined the two sections?" 2.MD.5

Missing measurements problems



What are the missing lengths of the third and fourth sides of the rectangle?

What are the missing lengths of each step and the bottom of the stairway?

These problems might be presented in the context of turtle geometry. Students work on paper to figure out how far the Logo turtle would have to travel to *finish* drawing the house (the remainder of the right side, and the bottom). They then type in Logo commands (e.g., for the rectangle, forward 40 right 90 fd 100 rt 90 fd 20 fd 20 rt 90 fd 100) to check their calculations (MP5).

[2.MD.6] Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

[2.MD.3] Estimate lengths using units of inches, feet, centimeters, and meters.

(e.g., a meter is about the distance from the floor to the top of a doorknob), instruction should also help children build understandings of scales and concepts of measurement into their estimation competencies. Although "guess and check" experiences can be useful, research suggests explicit teaching of estimation strategies (such as iteration of a mental image of the unit or comparison with a known measurement) and prompting students to learn reference or benchmark lengths (e.g., an inch-long piece of gum, a 6-inch dollar bill), order points along a continuum, and build up mental rulers.

Length measurement should also be used in other domains of mathematics, as well as in other subjects, such as science, and connections should be made where possible. For example, a line plot scale is just a ruler, usually with a non-standard unit of length. Teachers can ask students to discuss relationships they see between rulers and line plot scales. Data using length measures might be graphed (see example on pp. 8–9 of the Measurement Data Progression). Students could also graph the results of many students measuring the same object as precisely as possible (even involving halves or fourths of a unit) and discuss what the "real" measurement of the object might be. Emphasis on students solving real measurement problems, and, in so doing, building and iterating units, as well as units of units, helps students development strong concepts and skills. When conducted in this way, measurement tools and procedures become tools for mathematics and tools for thinking about mathematics.

Area and volume: Foundations To learn area (and, later, volume) concepts and skills meaningfully in later grades, students need to develop the ability known as *spatial structuring*. Students need to be able to see a rectangular region as decomposable into rows and columns of squares. This competence is discussed in detail in the Geometry Progression, but is mentioned here for two reasons. First, such spatial structuring precedes meaningful mathematical use of the structures, such as determining area or volume. Second, Grade 2 work in multiplication involves work with rectangular arrays,[2.G.2] and this work is an ideal context in which to simultaneously develop both arithmetical and spatial structuring foundations for later work with area.

[2.G.2] Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

Grade 3

Perimeter Third graders focus on solving real-world and mathematical problems involving perimeters of polygons.[3.MD.8] A perimeter is the boundary of a two-dimensional shape. For a polygon, the length of the perimeter is the sum of the lengths of the sides. Initially, it is useful to have sides marked with unit length marks, allowing students to count the unit lengths. Later, the lengths of the sides can be labeled with numerals. As with all length tasks, students need to count the length-units and not the end-points. Next, students learn to mark off unit lengths with a ruler and label the length of each side of the polygon. For rectangles, parallelograms, and regular polygons, students can discuss and justify faster ways to find the perimeter length than just adding all of the lengths (MP3). Rectangles and parallelograms have opposite sides of equal length, so students can double the lengths of adjacent sides and add those numbers or add lengths of two adjacent sides and double that number. A regular polygon has all sides of equal length, so its perimeter length is the product of one side length and the number of sides.

Perimeter problems for rectangles and parallelograms often give only the lengths of two adjacent sides or only show numbers for these sides in a drawing of the shape. The common error is to add just those two numbers. Having students first label the lengths of the other two sides as a reminder is helpful.

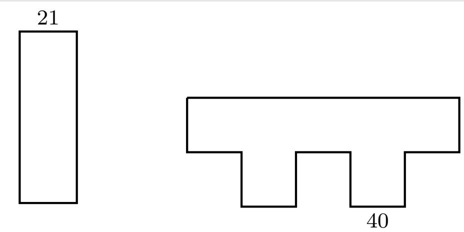
Students then find unknown side lengths in more difficult "missing measurements" problems and other types of perimeter problems.[3.MD.8]

Children learn to subdivide length-units. Making one's own ruler and marking halves and other partitions of the unit may be helpful in this regard. For example, children could fold a unit in halves, mark the fold as a half, and then continue to do so, to build fourths and eighths, discussing issues that arise. Such activities relate to fractions on the number line.[3.NF.2] Labeling all of the fractions can help students understand rulers marked with halves and fourths but not labeled with these fractions. Students also measure lengths using rulers marked with halves and fourths of an inch.[3.MD.4] They show these data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters (see the Measurement Data Progression, p. 10).

Understand concepts of area and relate area to multiplication and to addition Third graders focus on learning area. Students learn formulas to compute area, with those formulas based on, and summarizing, a firm conceptual foundation about what area is. Students need to learn to conceptualize area as the amount of two-dimensional space in a bounded region and to measure it by choosing a unit of area, often a square. A two-dimensional geometric figure that is covered by a certain number of squares without gaps

[3.MD.8] Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

Missing measurements and other perimeter problems



The perimeter of this rectangle is 168 length units. What are the lengths of the three unlabeled sides?

Assume all short segments are the same length and all angles are right

Compare these problems with the "missing measurements" problems of Grade 2.

Another type of perimeter problem is to draw a robot on squared grid paper that meets specific criteria. All the robot's body parts must be rectangles. The perimeter of the head might be 36 length-units, the body, 72; each arm, 24; and each leg, 72. Students are asked to provide a convincing argument that their robots meet these criteria (MP3). Next, students are asked to figure out the area of each of their body parts (in square units). These are discussed, with students led to reflect on the different areas that may be produced with rectangles of the same perimeter. These types of problems can be also presented as turtle geometry problems. Students create the commands on paper and then give their commands to the Logo turtle to check their calculations. For turtle length units, the perimeter of the head might be 300 length-units, the body, 600; each arm, 400; and each leg, 640.

[3.NF.2] Understand a fraction as a number on the number line; represent fractions on a number line diagram.

[3.MD.4] Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters.

or overlaps can be said to have an area of that number of square units.[3.MD.5]

Activities such as those in the Geometry Progression teach students to compose and decompose geometric regions. To begin an explicit focus on area, teachers might then ask students which of three rectangles covers the most area. Students may first solve the problem with decomposition (cutting and/or folding) and re-composition, and eventually analyses with area-units, by covering each with unit squares (tiles).[3.MD.5, 3.MD.6] Discussions should clearly distinguish the attribute of area from other attributes, notably length.

Students might then find the areas of other rectangles. As previously stated, students can be taught to multiply length measurements to find the area of a rectangular region. But, in order that they make sense of these quantities (MP2), they first learn to interpret measurement of rectangular regions as a multiplicative relationship of the number of square units in a row and the number of rows.[3.MD.7a] This relies on the development of spatial structuring.[MP7] To build from spatial structuring to understanding the number of area-units as the product of number of units in a row and number of rows, students might draw rectangular arrays of squares and learn to determine the number of squares in each row with increasingly sophisticated strategies, such as skip-counting the number in each row and eventually multiplying the number in each row by the number of rows (MP8). They learn to partition a rectangle into identical squares by anticipating the final structure and forming the array by drawing line segments to form rows and columns. They use skip counting and multiplication to determine the number of squares in the array.

Many activities that involve seeing and making arrays of squares to form a rectangle might be needed to build robust conceptions of a rectangular area structured into squares. One such activity is illustrated in the margin. In this progression, less sophisticated activities of this sort were suggested for earlier grades so that Grade 3 students begin with some experience.

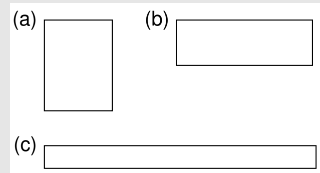
Students learn to understand and explain why multiplying the side lengths of a rectangle yields the same measurement of area as counting the number of tiles (with the same unit length) that fill the rectangle's interior (MP3).[3.MD.7a] For example, students might explain that one length tells how many unit squares in a row and the other length tells how many rows there are.

Students might then solve numerous problems that involve rectangles of different dimensions (e.g., designing a house with rooms that fit specific area criteria) to practice using multiplication to compute areas.[3.MD.7b] The areas involved should not all be rectangular, but decomposable into rectangles (e.g., an "L-shaped" room).[3.MD.7d]

Students also might solve problems such as finding all the rectangular regions with whole-number side lengths that have an area of 12 area-units, doing this later for larger rectangles (e.g., enclosing 24, 48, or 72 area-units), making sketches rather than drawing each

3.MD.5 Recognize area as an attribute of plane figures and understand concepts of area measurement.

Which rectangle covers the most area?



These rectangles are formed from unit squares (tiles students have used) although students are not informed of this or the rectangle's dimensions: (a) 4 by 3, (b) 2 by 6, and (c) 1 row of 12. Activity from Lehrer, et al., 1998, "Developing understanding of geometry and space in the primary grades," in R. Lehrer & D. Chazan (Eds.), *Designing Learning Environments for Developing Understanding of Geometry and Space*, Lawrence Erlbaum Associates.

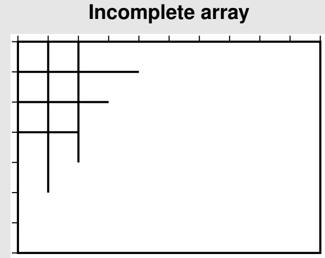
3.MD.5 Recognize area as an attribute of plane figures and understand concepts of area measurement.

3.MD.6 Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

3.MD.7a Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.

MP7 See the Geometry Progression

Incomplete array



To determine the area of this rectangular region, students might be encouraged to construct a row, corresponding to the indicated positions, then repeating that row to fill the region. Cutouts of strips of rows can help the needed spatial structuring and reduce the time needed to show a rectangle as rows or columns of squares. Drawing all of the squares can also be helpful, but it is slow for larger rectangles. Drawing the unit lengths on the opposite sides can help students see that joining opposite unit end-points will create the needed unit square grid.

3.MD.7b Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.

3.MD.7d Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.

square. They learn to justify their belief they have found all possible solutions (MP3).

Similarly using concrete objects or drawings, and their competence with composition and decomposition of shapes, spatial structuring, and addition of area measurements, students learn to investigate arithmetic properties using area models. For example, they learn to rotate rectangular arrays physically and mentally, understanding that their areas are preserved under rotation, and thus, for example, $4 \times 7 = 7 \times 4$, illustrating the commutative property of multiplication.[3.MD.7c] They also learn to understand and explain that the area of a rectangular region of, for example, 12 length-units by 5 length-units can be found either by multiplying 12×5, or by adding two products, e.g., 10×5 and 2×5, illustrating the distributive property.

3.MD.7c Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.

Recognize perimeter as an attribute of plane figures and distinguish between linear and area measures With strong and distinct concepts of both perimeter and area established, students can work on problems to differentiate their measures. For example, they can find and sketch rectangles with the same perimeter and different areas or with the same area and different perimeters and justify their claims (MP3).[3.MD.8] Differentiating perimeter from area is facilitated by having students draw congruent rectangles and measure, mark off, and label the unit lengths all around the perimeter on one rectangle, then do the same on the other rectangle but also draw the square units. This enables students to see the units involved in length and area and find patterns in finding the lengths and areas of non-square and square rectangles (MP7). Students can continue to describe and show the units involved in perimeter and area after they no longer need these .

3.MD.8 Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

Problem solving involving measurement and estimation of intervals of time, liquid volumes, and masses of objects Students in Grade 3 learn to solve a variety of problems involving measurement and such attributes as length and area, liquid volume, mass, and time.[3.MD.1, 3.MD.2] Many such problems support the Grade 3 emphasis on multiplication (see Table 1) and the mathematical practices of making sense of problems (MP1) and representing them with equations, drawings, or diagrams (MP4). Such work will involve units of mass such as the kilogram.

3.MD.1 Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

3.MD.2 Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l).[2] Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.[3]

Table 1: Multiplication and division situations for measurement

	Unknown Product $A \times B = \square$	Group Size Unknown $A \times \square = C$ and $C \div A = \square$	Number of Groups Unknown $\square \times B = C$ and $C \div B = \square$
Grouped Objects (Units of Units)	You need A lengths of string, each B inches long. How much string will you need altogether?	You have C inches of string, which you will cut into A equal pieces. How long will each piece of string be?	You have C inches of string, which you will cut into pieces that are B inches long. How many pieces of string will you have?
Arrays of Objects (Spatial Structuring)	What is the area of a A cm by B cm rectangle?	A rectangle has area C square centimeters. If one side is A cm long, how long is a side next to it?	A rectangle has area C square centimeters. If one side is B cm long, how long is a side next to it?
Compare	A rubber band is B cm long. How long will the rubber band be when it is stretched to be A times as long?	A rubber band is stretched to be C cm long and that is A times as long as it was at first. How long was the rubber band at first?	A rubber band was B cm long at first. Now it is stretched to be C cm long. How many times as long is the rubber band now as it was at first?

Adapted from box 2-4 of *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, National Research Council, 2009, pp. 32–33. Note that Grade 3 work does not include Compare problems with "times as much," see the Operations and Algebraic Thinking Progression, Table 3, also p. 29.

A few words on volume are relevant. Compared to the work in area, volume introduces more complexity, not only in adding a third dimension and thus presenting a significant challenge to students' spatial structuring, but also in the materials whose volumes are measured. These materials may be solid or fluid, so their volumes are generally measured with one of two methods, e.g., "packing" a right rectangular prism with cubic units or "filling" a shape such as a right circular cylinder. Liquid measurement, for many third graders, may be limited to a one-dimensional unit structure (i.e., simple iterative counting of height that is not processed as three-dimensional). Thus, third graders can learn to measure with liquid volume and to solve problems requiring the use of the four arithmetic operations, when liquid volumes are given in the same units throughout each problem. Because liquid measurement can be represented with one-dimensional scales, problems may be presented with drawings or diagrams, such as measurements on a beaker with a measurement scale in milliliters.

Grade 4

In Grade 4, students build on competencies in measurement and in building and relating units and units of units that they have developed in number, geometry, and geometric measurement.

Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit Fourth graders learn the relative sizes of measurement units within a system of measurement[4.MD.1] including:

> *length*: meter (m), kilometer (km), centimeter (cm), millimeter (mm); volume: liter (l), milliliter (ml, 1 cubic centimeter of water; a liter, then, is 1000 ml);
>
> *mass*: gram (g, about the weight of a cc of water), kilogram (kg); time: hour (hr), minute (min), second (sec).

For example, students develop benchmarks and mental images about a meter (e.g., about the height of a tall chair) and a kilometer (e.g., the length of 10 football fields including the end zones, or the distance a person might walk in about 12 minutes), and they also understand that "kilo" means a thousand, so 3000 m is equivalent to 3 km.

Expressing larger measurements in smaller units within the metric system is an opportunity to reinforce notions of place value. There are prefixes for multiples of the basic unit (meter or gram), although only a few (kilo-, centi-, and milli-) are in common use. Tables such as the one in the margin indicate the meanings of the prefixes by showing them in terms of the basic unit (in this case, meters). Such tables are an opportunity to develop or reinforce place value concepts and skills in measurement activities.

Relating units within the metric system is another opportunity to think about place value. For example, students might make a table that shows measurements of the same lengths in centimeters and meters.

Relating units within the traditional system provides an opportunity to engage in mathematical practices, especially "look for and make use of structure" (MP7) and "look for and express regularity in repeated reasoning" (MP8). For example, students might make a table that shows measurements of the same lengths in feet and inches.

Students also combine competencies from different domains as they solve measurement problems using all four arithmetic operations, addition, subtraction, multiplication, and division (see examples in Table 1).[4.MD.2] For example, "How many liters of juice does the class need to have at least 35 cups if each cup takes 225 ml?" Students may use tape or number line diagrams for solving such problems (MP1).

4.MD.1 Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.

Super- or subordinate unit	Length in terms of basic unit
kilometer	10^3 or 1000 meters
hectometer	10^2 or 100 meters
decameter	10^1 or 10 meters
meter	1 meter
decimeter	10^{-1} or $\frac{1}{10}$ meters
centimeter	10^{-2} or $\frac{1}{100}$ meters
millimeter	10^{-3} or $\frac{1}{1000}$ meters

Note the similarity to the structure of base-ten units and U.S. currency (see illustrations on p. 12 of the Number and Operations in Base Ten Progression).

Centimeter and meter equivalences

cm	m
100	1
200	2
300	3
500	
1000	

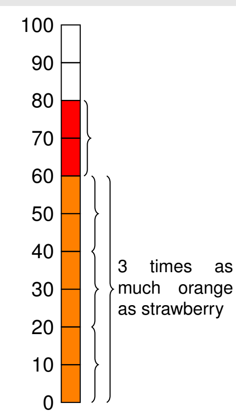
Foot and inch equivalences

feet	inches
0	0
1	12
2	24
3	

4.MD.2 Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

Using tape diagrams to solve word problems

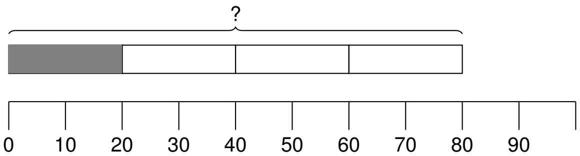
Lisa put two flavors of soda in a glass. There were 80 ml of soda in all. She put three times as much orange drink as strawberry. How many ml of orange did she put in?



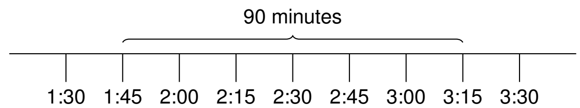
In this diagram, quantities are represented on a measurement scale.

Using number line diagrams to solve word problems

Juan spent 1/4 of his money on a game. The game cost $20. How much money did he have at first?



What time does Marla have to leave to be at her friend's house by a quarter after 3 if the trip takes 90 minutes?



Using a number line diagram to represent time is easier if students think of digital clocks rather than round clocks. In the latter case, placing the numbers on the number line involves considering movements of the hour and minute hands.

Students learn to consider perimeter and area of rectangles, begun in Grade 3, more abstractly (MP2). Based on work in previous grades with multiplication, spatially structuring arrays, and area, they abstract the formula for the area of a rectangle $A = l \times w$.•

Students generate and discuss advantages and disadvantages of various formulas for the perimeter length of a rectangle that is l units by w units.• Giving verbal summaries of these formulas is also helpful. For example, a verbal summary of the basic formula, $A = l + w + l + w$, is "add the lengths of all four sides." Specific numerical instances of other formulas or mental calculations for the perimeter of a rectangle can be seen as examples of the properties of operations, e.g., $2l + 2w = 2(l + w)$ illustrates the distributive property.

Perimeter problems often give only one length and one width, thus remembering the basic formula can help to prevent the usual error of only adding one length and one width. The formula $P = 2(l + w)$ emphasizes the step of multiplying the total of the given lengths by 2. Students can make a transition from showing all length units along the sides of a rectangle or all area units within (as in Grade 3, p. 18) by drawing a rectangle showing just parts of these as a reminder of which kind of unit is being used. Writing all of the lengths around a rectangle can also be useful. Discussions of formulas such as $P = 2l + 2w$, can note that unlike area formulas, perimeter formulas combine length measurements to yield a length measurement.

Such abstraction and use of formulas underscores the importance of distinguishing between area and perimeter in Grade 3[3.MD.8] and maintaining the distinction in Grade 4 and later grades, where rectangle perimeter and area problems may get more complex and problem solving can benefit from knowing or being able to rapidly remind oneself of how to find an area or perimeter. By repeatedly reasoning about how to calculate areas and perimeters of rectangles, students can come to see area and perimeter formulas as summaries of all such calculations (MP8).

• The formula is a generalization of the understanding, that, given a unit of length, a rectangle whose sides have length w units and l units, can be partitioned into w rows of unit squares with l squares in each row. The product $l \times w$ gives the number of unit squares in the partition, thus the area measurement is $l \times w$ square units. These square units are derived from the length unit.

• For example, $P = 2l + 2w$ has two multiplications and one addition, but $P = 2(l + w)$, which has one addition and one multiplication, involves fewer calculations. The latter formula is also useful when generating all possible rectangles with a given perimeter. The length and width vary across all possible pairs whose sum is half of the perimeter (e.g., for a perimeter of 20, the length and width are all of the pairs of numbers with sum 10).

3.MD.8 Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

Students learn to apply these understandings and formulas to the solution of real-world and mathematical problems.[4.MD.3] For example, they might be asked, "A rectangular garden has as an area of 80 square feet. It is 5 feet wide. How long is the garden?" Here, specifying the area and the width, creates an unknown factor problem (see Table 1). Similarly, students could solve perimeter problems that give the perimeter and the length of one side and ask the length of the adjacent side. Students could be challenged to solve multi-step problems such as the following. "A plan for a house includes rectangular room with an area of 60 square meters and a perimeter of 32 meters. What are the length and the width of the room?"

4.MD.3 Apply the area and perimeter formulas for rectangles in real world and mathematical problems.

In Grade 4 and beyond, the mental visual images for perimeter and area from Grade 3 can support students in problem solving with these concepts. When engaging in the mathematical practice of reasoning abstractly and quantitatively (MP2) in work with area and perimeter, students think of the situation and perhaps make a drawing. Then they recreate the "formula" with specific numbers and one unknown number as a situation equation for this particular numerical situation.• "Apply the formula" does not mean write down a memorized formula and put in known values because at Grade 4 students do not evaluate expressions (they begin this type of work in Grade 6). In Grade 4, working with perimeter and area of rectangles is still grounded in specific visualizations and numbers. These numbers can now be any of the numbers used in Grade 4 (for addition and subtraction for perimeter and for multiplication and division for area).[4.NBT.4, 4.NF.3d, 4.OA.4] By repeatedly reasoning about constructing situation equations for perimeter and area involving specific numbers and an unknown number, students will build a foundation for applying area, perimeter, and other formulas by substituting specific values for the variables in later grades.

• "Situation equation" refers to the idea that the student constructs an equation as a representation of a situation rather than identifying the situation as an example of a familiar equation.

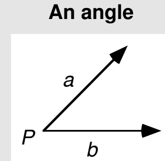
4.NBT.4 Fluently add and subtract multi-digit whole numbers using the standard algorithm.

4.NF.3d Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem.

4.OA.4 Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.

Understand concepts of angle and measure angles Angle measure is a "turning point" in the study of geometry. Students often find angles and angle measure to be difficult concepts to learn, but that learning allows them to engage in interesting and important mathematics. An *angle* is the union of two rays, a and b, with the same initial point P. The rays can be made to coincide by rotating one to the other about P; this rotation determines the size of the angle between a and b. The rays are sometimes called the *sides* of the angles.

An angle



P is called the *vertex* of the angle and the rays a and b are called the *arms*.

Another way of saying this is that each ray determines a direction and the angle size measures the change from one direction to the other. (This illustrates how angle measure is related to the concepts of parallel and perpendicular lines in Grade 4 geometry.) A clockwise rotation is considered positive in surveying or turtle geometry; but a counterclockwise rotation is considered positive in Euclidean geometry. Angles are measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through $\frac{1}{360}$ of a circle is called a "one-degree angle," and degrees are the unit used to measure angles in elementary school. A full rotation is thus 360°.

Two angles are called *complementary* if their measurements have the sum of 90°. Two angles are called *supplementary* if their measurements have the sum of 180°. Two angles with the same vertex that overlap only at a boundary (i.e., share a side) are called *adjacent angles*.

Like length, area, and volume, angle measure is additive: The sum of the measurements of *adjacent angles* is the measurement of the angle formed by their union. This leads to other important properties. If a right angle is decomposed into two adjacent angles, the sum is 90°, thus they are complementary. Two adjacent angles that compose a "straight angle" of 180° must be supplementary. In some situations (see margin), such properties allow logical progressions of statements (MP3).

As with all measureable attributes, students must first recognize the attribute of angle measure, and distinguish it from other attributes. This may not appear too difficult, as the measure of angles and rotations appears to knowledgeable adults as quite different than attributes such as length and area. However, the unique nature of angle size leads many students to initially confuse angle measure with other, more familiar, attributes. Even in contexts designed to evoke a dynamic image of turning, such as hinges or doors, many students use the length between the endpoints, thus teachers find it useful to repeatedly discuss such cognitive "traps."

As with other concepts (e.g., see the Geometry Progression), students need varied examples and explicit discussions to avoid learning limited ideas about measuring angles (e.g., misconceptions that a right angle is an angle that points to the right, or two right angles represented with different orientations are not equal in measure). If examples and tasks are not varied, students can develop incomplete and inaccurate notions. For example, some come to associate all slanted lines with 45° measures and horizontal and vertical lines with measures of 90°. Others believe angles can be "read off" a protractor in "standard" position, that is, a base is horizontal, even if neither arm of the angle is horizontal. Measuring and then sketching many angles with no horizontal or vertical arms,[4.MD.6] perhaps initially using circular 360° protractors, can help students avoid such limited conceptions.

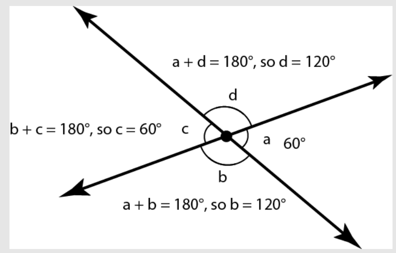
As with length, area, and volume, children need to understand equal partitioning and unit iteration to understand angle and turn measure. Whether defined as more statically as the measure of the figure formed by the intersection of two rays or as turning, having a given angle measure involves a relationship between components of plane figures and therefore is a *property* (see the Overview in the Geometry Progression).[4.G.2]

Given the complexity of angles and angle measure, it is unsur-

An angle

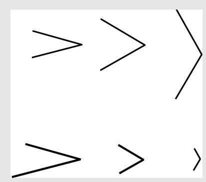
name	measurement
right angle	90°
straight angle	180°
acute angle	between 0 and 90°
obtuse angle	between 90° and 180°
reflex angle	between 180° and 360°

Angles created by the intersection of two lines



When two lines intersect, they form four angles. If the measurement of one is known (e.g., angle a is 60°), the measurement of the other three can be determined.

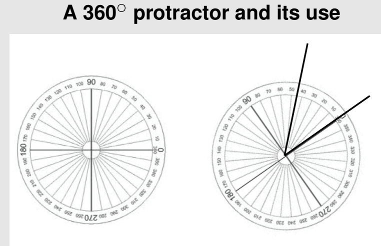
Two representations of three angles



Initially, some students may correctly compare angle sizes only if all the line segments are the same length (as shown in the top row). If the lengths of the line segments are different (as shown in the bottom row), these students base their judgments on the lengths of the segments, the distances between their endpoints, or even the area of the triangles determined by the drawn arms. They believe that the angles in the bottom row decrease in size from left to right, although they have, respectively, the same angle measurements as those in the top row.

4.MD.6 Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.

A 360° protractor and its use



The figure on the right shows a protractor being used to measure a 45° angle. The protractor is placed so that one side of the angle lies on the line corresponding to 0° on the protractor and the other side of the angle is located by a clockwise rotation from that line.

4.G.2 Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

prising that students in the early and elementary grades often form separate concepts of angles as figures and turns, and may have separate notions for different turn contexts (e.g., unlimited rotation as a fan vs. a hinge) and for various "bends."

However, students can develop more accurate and useful angle and angle measure concepts if presented with angles in a variety of situations. They learn to find the common features of superficially different situations such as turns in navigation, slopes, bends, corners, and openings. With guidance, they learn to represent an angle in any of these contexts as two rays, even when both rays are not explicitly represented in the context; for example, the horizontal or vertical in situations that involve slope (e.g., roads or ramps), or the angle determined by looking up from the horizon to a tree- or mountain-top. Eventually they abstract the common attributes of the situations as angles (which are represented with rays and a vertex, MP4) and angle measurements (MP2). To accomplish the latter, students integrate turns, and a general, dynamic understanding of angle measure-as-rotation, into their understandings of angles-as-objects. Computer manipulatives and tools can help children bring such a dynamic concept of angle measure to an explicit level of awareness. For example, dynamic geometry environments can provide multiple linked representations, such as a screen drawing that students can "drag" which is connected to a numerical representation of angle size. Games based on similar notions are particularly effective when students manipulate not the arms of the angle itself, but a representation of rotation (a small circular diagram with radii that, when manipulated, change the size of the target angle turned).

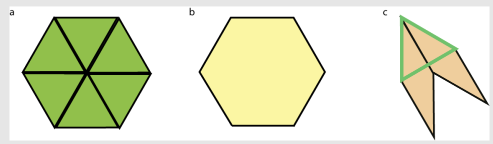
Students with an accurate conception of angle can recognize that angle measure is *additive*.[4.MD.7] As with length, area, and volume, when an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Students can then solve interesting and challenging addition and subtraction problems to find the measurements of unknown angles on a diagram in real world and mathematical problems. For example, they can find the measurements of angles formed a pair of intersecting lines, as illustrated above, or given a diagram showing the measurement of one angle, find the measurement of its complement. They can use a protractor to check, not to check their reasoning, but to ensure that they develop full understanding of the mathematics and mental images for important benchmark angles (e.g., 30°, 45°, 60°, and 90°).

Such reasoning can be challenged with many situations as illustrated in the margin.

Similar activities can be done with drawings of shapes using right angles and half of a right angle to develop the important benchmarks of 90° and 45°.

Missing measures can also be done in the turtle geometry context, building on the previous work. Note that unguided use of Logo's turtle geometry does not necessary develop strong angle

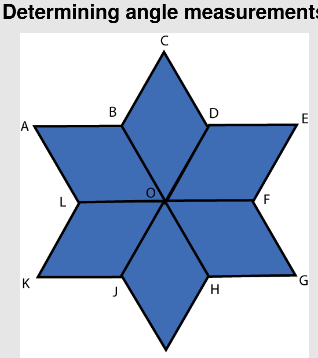
Determining angles in pattern blocks



Students might determine all the angles in the common "pattern block" shape set based on equilateral triangles. Placing six equilateral triangles so that they share a common vertex (as shown in part a), students can figure out that because the sum of the angles at this vertex is 360°, each angle which shares this vertex must have measure 60°. Because they are congruent, all the angles of the equilateral triangles must have measure 60° (again, to ensure they develop a firm foundation, students can verify these for themselves with a protractor). Because each angle of the regular hexagon (part b) is composed of two angles from equilateral triangles, the hexagon's angles each measure 120°. Similarly, in a pattern block set, two of the smaller angles from tan rhombi compose an equilateral triangle's angle, so each of the smaller rhombus angles has measure 30°.

4.MD.7 Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.

Determining angle measurements

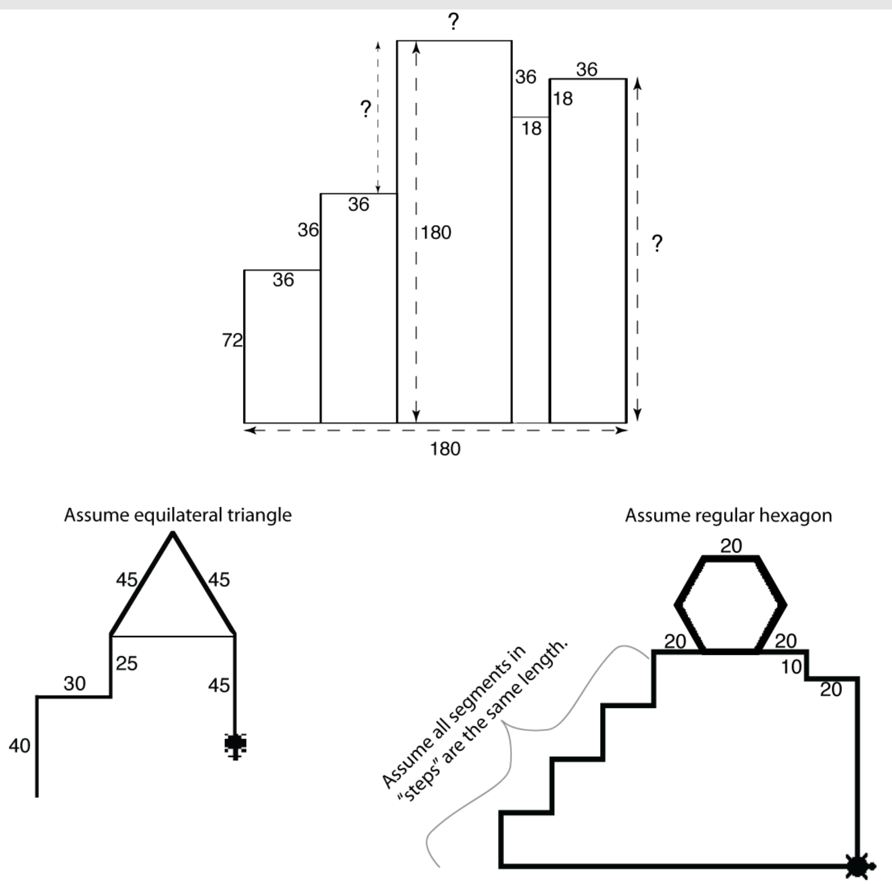


Students might be asked to determine the measurements of the following angles:
$\angle BOD$
$\angle BOF$
$\angle ODE$
$\angle CDE$
$\angle CDJ$
$\angle BHG$

concepts. However, if teachers emphasize mathematical tasks and, within those tasks, the difference between the angle of rotation the turtle makes (in a polygon, the external angle) and the angle formed (internal angle) and integrates the two, students can develop accurate and comprehensive understandings of angle measure. For example, what series of commands would produce a square? How many degrees would the turtle turn? What is the measure of the resulting angle? What would be the commands for an equilateral triangle? How many degrees would the turtle turn? What is the measure of the resulting angle? Such questions help to connect what are often initially isolated ideas about angle conceptions.

These understandings support students in finding all the missing length and angle measures in situations such as the examples in the margin (compare to the missing measures problems Grade 2 and Grade 3).

Missing measures: Length (top) and length and angle (turn)



Students are asked to determine the missing lengths. They might first work on paper to figure out how far the Logo turtle would have to travel to *finish* drawing the house, then type in Logo commands to verify their reasoning and calculations.

Grade 5

Convert like measurement units within a given measurement system In Grade 5, students extend their abilities from Grade 4 to express measurements in larger or smaller units within a measurement system.[4.MD.1, 5.MD.1] This is an excellent opportunity to reinforce notions of place value for whole numbers and decimals, and connection between fractions and decimals (e.g., $2\frac{1}{2}$ meters can be expressed as 2.5 meters or 250 centimeters). For example, building on the table from Grade 4, Grade 5 students might complete a table of equivalent measurements in feet and inches.

Grade 5 students also learn and use such conversions in solving multi-step, real world problems (see example in the margin).

Understand concepts of volume and relate volume to multiplication and to addition The major emphasis for measurement in Grade 5 is volume. Volume not only introduces a third dimension and thus a significant challenge to students' spatial structuring, but also complexity in the nature of the materials measured. That is, solid units are "packed," such as cubes in a three-dimensional array, whereas a liquid "fills" three-dimensional space, taking the shape of the container. As noted earlier (see Overview, also Grades 1 and 3), the unit structure for liquid measurement may be psychologically one-dimensional for some students.

"Packing" volume is more difficult than iterating a unit to measure length and measuring area by tiling. Students learn about a unit of volume, such as a cube with a side length of 1 unit, called a unit cube.[5.MD.3] They pack cubes (without gaps) into right rectangular prisms and count the cubes to determine the volume or build right rectangular prisms from cubes and see the layers as they build.[5.MD.4] They can use the results to compare the volume of right rectangular prisms that have different dimensions. Such experiences enable students to extend their spatial structuring from two to three dimensions (see the Geometry Progression). That is, they learn to both mentally decompose and recompose a right rectangular prism built from cubes into layers, each of which is composed of rows and columns. That is, given the prism, they have to be able to decompose it, understanding that it can be partitioned into layers, and each layer partitioned into rows, and each row into cubes. They also have to be able to compose such as structure, multiplicatively, back into higher units. That is, they eventually learn to conceptualize a layer as a unit that itself is composed of units of units—rows, each row composed of individual cubes—and they iterate that structure. Thus, they might predict the number of cubes that will be needed to fill a box given the net of the box.

Another complexity of volume is the connection between "packing" and "filling." Often, for example, students will respond that a box can be filled with 24 centimeter cubes, or build a structure of 24 cubes, and still think of the 24 as individual, often discrete, not

[4.MD.1] Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.

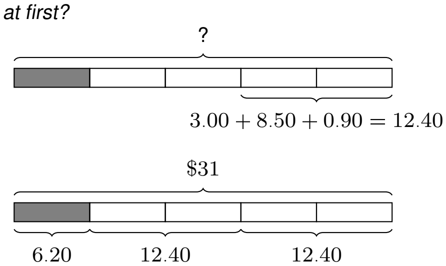
[5.MD.1] Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

Feet	Inches
0	0
	1
	2
	3

In Grade 6, this table can be discussed in terms of ratios and proportional relationships (see the Ratio and Proportion Progression). In Grade 5, however, the main focus is on arriving at the measurements that generate the table.

Multi-step problem with unit conversion

Kumi spent a fifth of her money on lunch. She then spent half of what remained. She bought a card game for $3, a book for $8.50, and candy for 90 cents. How much money did she have at first?

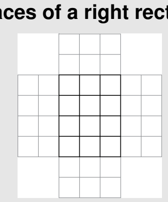


Students can use tape diagrams to represent problems that involve conversion of units, drawing diagrams of important features and relationships (MP1).

[5.MD.3] Recognize volume as an attribute of solid figures and understand concepts of volume measurement.

[5.MD.4] Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.

Net for five faces of a right rectangular prism



Students are given a net and asked to predict the number of cubes required to fill the container formed by the net. In such tasks, students may initially count single cubes or repeatedly add the number of cubes in a row to determine the number in each layer, and repeatedly add the number in each layer to find the total number of unit cubes. In folding the net to make the shape, students can see how the side rectangles fit together and determine the number of layers.

necessarily *units of volume*. They may, for example, not respond confidently and correctly when asked to fill a graduated cylinder marked in cubic centimeters with the amount of liquid that would fill the box. That is, they have not yet connected their ideas about filling volume with those concerning packing volume. Students learn to move between these conceptions, e.g., using the same container, both filling (from a graduated cylinder marked in ml or cc) and packing (with cubes that are each 1 cm^3). Comparing and discussing the volume-units and what they represent can help students learn a general, complete, and interconnected conceptualization of volume as filling three-dimensional space.

Students then learn to determine the volumes of several right rectangular prisms, using cubic centimeters, cubic inches, and cubic feet. With guidance, they learn to increasingly apply multiplicative reasoning to determine volumes, looking for and making use of structure (MP7). That is, they understand that multiplying the length times the width of a right rectangular prism can be viewed as determining how many cubes would be in each layer if the prism were packed with or built up from unit cubes.[5.MD.5a] They also learn that the height of the prism tells how many layers would fit in the prism. That is, they understand that volume is a derived attribute that, once a length unit is specified, can be computed as the product of three length measurements or as the product of one area and one length measurement.

5.MD.5a Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.

Then, students can learn the formulas $V = l \times w \times h$ and $V = B \times h$ for right rectangular prisms as efficient methods for computing volume, maintaining the connection between these methods and their previous work with computing the number of unit cubes that pack a right rectangular prism.[5.MD.5b] They use these competencies to find the volumes of right rectangular prisms with edges whose lengths are whole numbers and solve real-world and mathematical problems involving such prisms.

5.MD.5b Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real world and mathematical problems.

Students also recognize that volume is additive (see Overview) and they find the total volume of solid figures composed of two right rectangular prisms.[5.MD.5c] For example, students might design a science station for the ocean floor that is composed of several rooms that are right rectangular prisms and that meet a set criterion specifying the total volume of the station. They draw their station (e.g., using an isometric grid, MP7) and justify how their design meets the criterion (MP1).

5.MD.5c Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.

Where the Geometric Measurement Progression is heading

Connection to Geometry In Grade 6, students build on their understanding of length, area, and volume measurement, learning to how to compute areas of right triangles and other special figures and volumes of right rectangular prisms that do not have measurements given in whole numbers. To do this, they use dissection arguments. These rely on the understanding that area and volume measures are additive, together with decomposition of plane and solid shapes (see the K–5 Geometry Progression) into shapes whose measurements students already know how to compute (MP1, MP7). In Grade 7, they use their understanding of length and area in learning and using formulas for the circumference and area of circles. In Grade 8, they use their understanding of volume in learning and using formulas for the volumes of cones, cylinders, and spheres. In high school, students learn formulas for volumes of pyramids and revisit the formulas from Grades 7 and 8, explaining them with dissection arguments, Cavalieri's principle, and informal limit arguments.

Connection to the Number System In Grade 6, understanding of length-units and spatial structuring comes into play as students learn to plot points in the coordinate plane.

Connection to Ratio and Proportion Students use their knowledge of measurement and units of measurement in Grades 6–8, coming to see conversions between two units of measurement as describing proportional relationships.

Progressions for the Common Core State Standards in Mathematics (draft)

26 December 2011

6–8 Statistics and Probability

Overview

In Grade 6, students build on the knowledge and experiences in data analysis developed in earlier grades (see K-3 Categorical Data Progression and Grades 2-5 Measurement Progression). They develop a deeper understanding of variability and more precise descriptions of data distributions, using numerical measures of center and spread, and terms such as cluster, peak, gap, symmetry, skew, and outlier. They begin to use histograms and box plots to represent and analyze data distributions. As in earlier grades, students view statistical reasoning as a four-step investigative process:

- Formulate questions that can be answered with data
- Design and use a plan to collect relevant data
- Analyze the data with appropriate methods
- Interpret results and draw valid conclusions from the data that relate to the questions posed.

Such investigations involve making sense of practical problems by turning them into statistical investigations (MP1); moving from context to abstraction and back to context (MP2); repeating the process of statistical reasoning in a variety of contexts (MP8).

In Grade 7, students move from concentrating on analysis of data to production of data, understanding that good answers to statistical questions depend upon a good plan for collecting data relevant to the questions of interest. Because statistically sound data production is based on random sampling, a probabilistic concept, students must develop some knowledge of probability before launching into sampling. Their introduction to probability is based on seeing probabilities of chance events as long-run relative frequencies of their occurrence, and many opportunities to develop the connection between theoretical probability models and empirical probability approximations. This connection forms the basis of statistical inference.

With random sampling as the key to collecting good data, students begin to differentiate between the variability in a sample and the variability inherent in a statistic computed from a sample when samples are repeatedly selected from the same population. This understanding of variability allows them to make rational decisions, say, about how different a proportion of "successes" in a sample is likely to be from the proportion of "successes" in the population or whether medians of samples from two populations provide convincing evidence that the medians of the two populations also differ.

Until Grade 8, almost all of students' statistical topics and investigations have dealt with univariate data, e.g., collections of counts or measurements of one characteristic. Eighth graders apply their experience with the coordinate plane and linear functions in the study of association between two variables related to a question of interest. As in the univariate case, analysis of bivariate measurement data graphed on a scatterplot proceeds by describing shape, center, and spread. But now "shape" refers to a cloud of points on a plane, "center" refers to a line drawn through the cloud that captures the essence of its shape, and "spread" refers to how far the data points stray from this central line. Students extend their understanding of "cluster" and "outlier" from univariate data to bivariate data. They summarize bivariate categorical data using two-way tables of counts and/or proportions, and examine these for patterns of association.

Grade 6

Develop understanding of statistical variability Statistical investigations begin with a question, and students now see that answers to such questions always involve variability in the data collected to answer them.[6.SP.1] Variability may seem large, as in the selling prices of houses, or small, as in repeated measurements on the diameter of a tennis ball, but it is important to interpret variability in terms of the situation under study, the question being asked, and other aspects of the data distribution (MP2). A collection of test scores that vary only about three percentage points from 90% as compared to scores that vary ten points from 70% lead to quite different interpretations by the teacher. Test scores varying by only three points is often a good situation. But what about the same phenomenon in a different context: percentage of active ingredient in a prescription drug varying by three percentage points from order to order?

Working with counts or measurements, students display data with the dot plots (sometimes called line plots) that they used in earlier grades. New at Grade 6 is the use of histograms, which are especially appropriate for large data sets.

Students extend their knowledge of symmetric shapes,[4.G.3] to describe data displayed in dot plots and histograms in terms of symmetry. They identify clusters, peaks, and gaps, recognizing common shapes[6.SP.2] and patterns in these displays of data distributions (MP7).

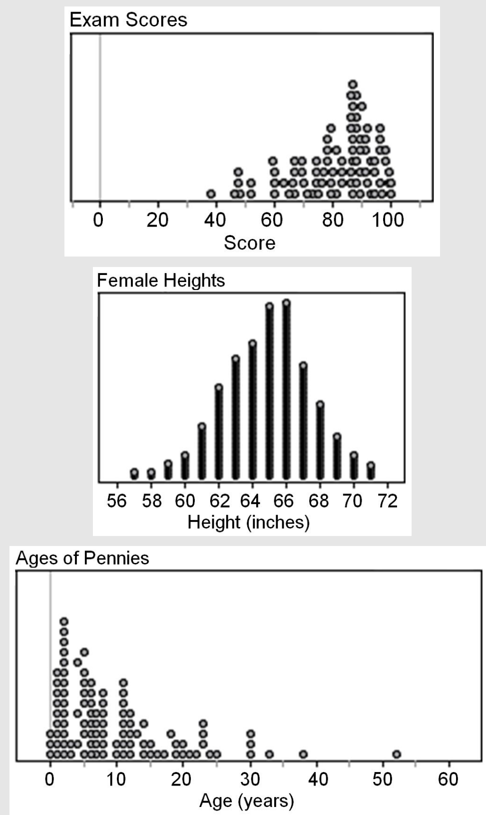
A major focus of Grade 6 is characterization of data distributions by measures of center and spread.[6.SP.2,6.SP.3] To be useful, center and spread must have well-defined numerical descriptions that are commonly understood by those using the results of a statistical investigation. The simpler ones to calculate and interpret are those based on counting. In that spirit, center is measured by the *median*, a number arrived at by counting to the middle of an ordered array of numerical data. When the number of data points is odd, the median is the middle value. When the number of data points is even, the median is the average of the two middle values. *Quartiles*, the medians of the lower and upper halves of the ordered data values, mark off the middle 50% of the data values and, thus, provide information on the spread of the data.[1] The distance between the first and third quartiles, the *interquartile range* (IQR), is a single number summary that serves as a very useful measure of variability.[6.SP.3]

Plotting the extreme values, the quartiles, and the median (the *five-number summary*) on a number line diagram, leads to the *box plot*, a concise way of representing the main features of a data dis-

[1]Different methods for computing quartiles are in use. The Standards uses the method which excludes the median to create two halves when the number of data points is odd. See Langford, "Quartiles in Elementary Statistics," *Journal of Statistics Education*, 2006, for a description of the different methods used by statisticians and statistical software.

[6.SP.1] Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers.

Dot plots: Skewed left, symmetric, skewed right

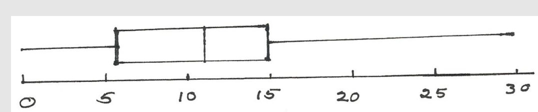


Students distinguish between dot plots showing distributions which are skewed left (skewed toward smaller values), approximately symmetric, and skewed right (skewed toward larger values). The plots show scores on a math exam, heights of 1,000 females with ages from 18 to 24, ages of 100 pennies in a sample collected from students.

[4.G.3] Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

[6.SP.2] Understand that a set of data collected to answer a statistical question has a distribution which can be described by its center, spread, and overall shape.

Box plot



For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 30}, the median is 11 (from the average of the two middle values 10 and 12), the interquartile range is $15 - 6 = 9$, and the extreme values are 1 and 30.

tribution.• Box plots are particularly well suited for comparing two or more data sets, such as the lengths of mung bean sprouts for plants with no direct sunlight versus the lengths for plants with four hours of direct sunlight per day.[6.SP.4]

Students use their knowledge[6.NS.2,6.NS.3] of division, fractions, and decimals in computing a new measure of center—the *arithmetic mean*, often simply called the *mean*. They see the mean as a "leveling out" of the data in the sense of a unit rate (see Ratio and Proportion Progression). In this "leveling out" interpretation, the mean is often called the "average" and can be considered in terms of "fair share." For example, if it costs a total of $40 for five students to go to lunch together and they decide to pay equal shares of the cost, then each student's share is $8.00. Students recognize the mean as a convenient summary statistic that is used extensively in the world around them, such as average score on an exam, mean temperature for the day, average height and weight of a person of their age, and so on.

Students also learn some of the subtleties of working with the mean, such as its sensitivity to changes in data values and its tendency to be pulled toward an extreme value, much more so than the median. Students gain experience in deciding whether the mean or the median is the better measure of center in the context of the question posed. Which measure will tend to be closer to where the data on prices of a new pair of jeans actually cluster? Why does your teacher report the mean score on the last exam? Why does your science teacher say, "Take three measurements and report the average?"

For distributions in which the mean is the better measure of center, variation is commonly measured in terms of how far the data values deviate from the mean. Students calculate how far each value is above or below the mean, and these deviations from the mean are the first step in building a measure of variation based on spread to either side of center. The average of the deviations is always zero, but averaging the absolute values of the deviations leads to a measure of variation that is useful in characterizing the spread of a data distribution and in comparing distributions. This measure is called the *mean absolute deviation*, or MAD. Exploring variation with the MAD sets the stage for introducing the standard deviation in high school.

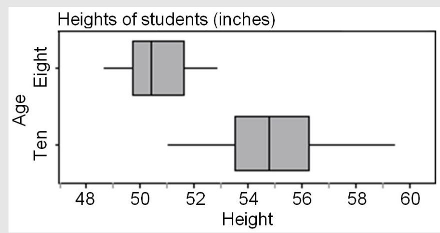
Summarize and describe distributions "How many text messages do middle school students send in a typical day?" Data obtained from a sample of students may have a distribution with a few very large values, showing a "long tail" in the direction of the larger values. Students realize that the mean may not represent the largest cluster of data points, and that the median is a more useful measure of center. In like fashion, the IQR is a more useful measure of spread, giving the spread of the middle 50% of the data points.

6.SP.3 Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.

• "Box plot" is also sometimes written "boxplot." Because of the different methods for computing quartiles and other different conventions, there are different kinds of box plots in use. Box plots created from the five-number summary do not show points detached from the remainder of the diagram. However, box plots generated with statistical software may display these features.

6.SP.4 Display numerical data in plots on a number line, including dot plots, histograms, and box plots.

Comparing distributions with box plots

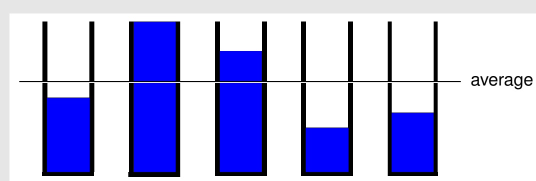


In Grade 6, box plots can be used to analyze the data from Example 2 of the Measurement Data Progression. Sixth graders can give more precise answers in terms of center and spread to questions asked at earlier grades. "Describe the key differences between the heights of these two age groups. What would you choose as the typical height of an eight-year-old? A ten-year-old? What would you say is the typical number of inches of growth from age eight to age ten?"

6.NS.2 Fluently divide multi-digit numbers using the standard algorithm.

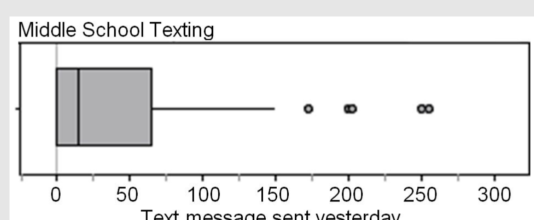
6.NS.3 Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

Average as a "leveling out"



As mentioned in the Grades 2-5 Measurement Data Progression, students in Grade 5 might find the amount of liquid each cylinder would contain if the total amount in all the cylinders were redistributed equally. In Grade 6, students are able to view the amount in each cylinder after redistribution as equal to the mean of the five original amounts.

Middle School Texting

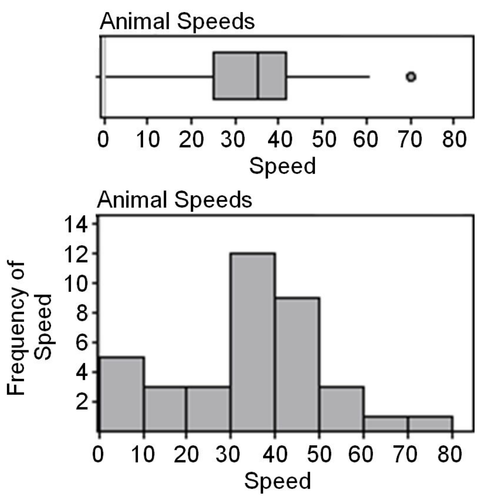


The 37 animal speeds shown in the margin can be used to illustrate summarizing a distribution.[6.SP.5a–c] According to the source, "Most of the following measurements are for maximum speeds over approximate quarter-mile distances. Exceptions—which are included to give a wide range of animals—are the lion and elephant, whose speeds were clocked in the act of charging; the whippet, which was timed over a 200-yard course; the cheetah over a 100-yard distance; humans for a 15-yard segment of a 100-yard run; and the black mamba snake, six-lined race runner, spider, giant tortoise, three-toed sloth, . . . , which were measured over various small distances." Understanding that it is difficult to measure speeds of wild animals, does this description raise any questions about whether or not this is a fair comparison of the speeds?

Moving ahead with the analysis, students will notice that the distribution is not symmetric, but the lack of symmetry is mild. It is most appropriate to measure center with the median of 35 mph and spread with the IQR of $42 - 25 = 17$. That makes the cheetah an outlier with respect to speed, but notice again the description of how this speed was measured. If the garden snail with a speed of 0.03 mph is added to the data set, then cheetah is no longer considered an outlier. Why is that?

Because the lack of symmetry is not severe, the mean (32.15 mph) is close to the median and the MAD (12.56 mph) is a reasonable measure of typical variation from the mean, as about 57% of the data values lie within one MAD of the mean, an interval from about 19.6 mph to 44.7 mph.

Box plot and histogram of 37 animal speeds



Note that the isolated point (the extreme value of 70 mph) has been generated by the software used to produce the box plot. The mild lack of symmetry can be seen in the box plot in the median (slightly off-center in the box) and in the slightly different lengths of the "whiskers." The geometric shape made by the histogram also shows mild lack of symmetry.

6.SP.5a Summarize numerical data sets in relation to their context, such as by:

a Reporting the number of observations.

6.SP.5b Summarize numerical data sets in relation to their context, such as by:

b Describing the nature of the attribute under investigation, including how it was measured and its units of measurement.

6.SP.5c Summarize numerical data sets in relation to their context, such as by:

c Giving quantitative measures of center (median and/or mean) and variability (interquartile range and/or mean absolute deviation), as well as describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered.

Table of 37 animal speeds

Animal	Speed (mph)
Cheetah	70.00
Pronghorn antelope	61.00
Lion	50.00
Thomson's gazelle	50.00
Wildebeest	50.00
Quarter horse	47.50
Cape hunting dog	45.00
Elk	45.00
Coyote	43.00
Gray fox	42.00
Hyena	40.00
Ostrich	40.00
Zebra	40.00
Mongolian wild ass	40.00
Greyhound	39.35
Whippet	35.50
Jackal	35.00
Mule deer	35.00
Rabbit (domestic)	35.00
Giraffe	32.00
Reindeer	32.00
Cat (domestic)	30.00
Kangaroo	30.00
Grizzly bear	30.00
Wart hog	30.00
White-tailed deer	30.00
Human	27.89
Elephant	25.00
Black mamba snake	20.00
Six-lined race runner	18.00
Squirrel	12.00
Pig (domestic)	11.00
Chicken	9.00
House mouse	8.00
Spider (Tegenearia atrica)	1.17
Giant tortoise	0.17
Three-toed sloth	0.15

Source: `factmonster.com/ipka/A0004737.html`

Grade 7

Chance processes and probability models In Grade 7, students build their understanding of probability on a relative frequency view of the subject, examining the proportion of "successes" in a chance process—one involving repeated observations of random outcomes of a given event, such as a series of coin tosses. "What is my chance of getting the correct answer to the next multiple choice question?" is not a probability question in the relative frequency sense. "What is my chance of getting the correct answer to the next multiple choice question *if I make a random guess among the four choices?*" is a probability question because the student could set up an experiment of multiple trials to approximate the relative frequency of the outcome.• And two students doing the same experiment will get nearly the same approximation. These important points are often overlooked in discussions of probability.[7.SP.5]

• Note the connection with MP6. Including the stipulation "if I make a random guess among the four choices" makes the question precise enough to be answered with the methods discussed for this grade.

7.SP.5 Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around 1/2 indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.

Students begin by relating probability to the long-run (more than five or ten trials) relative frequency of a chance event, using coins, number cubes, cards, spinners, bead bags, and so on. Hands-on activities with students collecting the data on probability experiments are critically important, but once the connection between observed relative frequency and theoretical probability is clear, they can move to simulating probability experiments via technology (graphing calculators or computers).

It must be understood that the connection between relative frequency and probability goes two ways. If you know the structure of the generating mechanism (e.g., a bag with known numbers of red and white chips), you can anticipate the relative frequencies of a series of random selections (with replacement) from the bag. If you do not know the structure (e.g., the bag has unknown numbers of red and white chips), you can approximate it by making a series of random selections and recording the relative frequencies.[7.SP.6] This simple idea, obvious to the experienced, is essential and not obvious at all to the novice.• The first type of situation, in which the structure is known, leads to "probability"; the second, in which the structure is unknown, leads to "statistics."

7.SP.6 Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability.

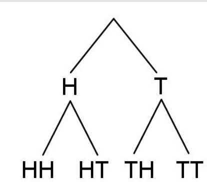
• Examples of student strategies for generalizing from the relative frequency in the simplest case (one sample) to the relative frequency in the whole population are given in the Ratio and Proportional Relationship Progression, p. 11.

A *probability model* provides a probability for each possible non-overlapping outcome for a chance process so that the total probability over all such outcomes is unity. The collection of all possible individual outcomes is known as the *sample space* for the model. For example, the sample space for the toss of two coins (fair or not) is often written as {TT, HT, TH, HH}. The probabilities of the model can be either *theoretical* (based on the structure of the process and its outcomes) or *empirical* (based on observed data generated by the process). In the toss of two balanced coins, the four outcomes of the sample space are given equal theoretical probabilities of $\frac{1}{4}$ because of the symmetry of the process—because the coins are balanced, an outcome of heads is just as likely as an outcome of tails. Randomly selecting a name from a list of ten students also leads to equally

Different representations of a sample space

HH
HT
TH
TT

	H	T
H	HH	HT
T	TH	TT



All the possible outcomes of the toss of two coins can be represented as an organized list, table, or tree diagram. The sample space becomes a probability model when a probability for each simple event is specified.

likely outcomes with probability 0.10 that a given student's name will be selected.[7.SP.7a] If there are exactly four seventh graders on the list, the chance of selecting a seventh grader's name is 0.40. On the other hand, the probability of a tossed thumbtack landing point up is not necessarily $\frac{1}{2}$ just because there are two possible outcomes; these outcomes may not be equally likely and an empirical answer must be found be tossing the tack and collecting data.[7.SP.7b]

The product rule for counting outcomes for chance events should be used in finite situations like tossing two or three coins or rolling two number cubes. There is no need to go to more formal rules for permutations and combinations at this level. Students should gain experience in the use of diagrams, especially trees and tables, as the basis for organized counting of possible outcomes from chance processes.[7.SP.8] For example, the 36 equally likely (theoretical probability) outcomes from the toss of a pair of number cubes are most easily listed on a two-way table. An archived table of census data can be used to approximate the (empirical) probability that a randomly selected Florida resident will be Hispanic.

After the basics of probability are understood, students should experience setting up a model and using simulation (by hand or with technology) to collect data and estimate probabilities for a real situation that is sufficiently complex that the theoretical probabilities are not obvious. For example, suppose, over many years of records, a river generates a spring flood about 40% of the time. Based on these records, what is the chance that it will flood for at least three years in a row sometime during the next five years?[7.SP.8c]

Random sampling In earlier grades students have been using data, both categorical and measurement, to answer simple statistical questions, but have paid little attention to how the data were selected. A primary focus for Grade 7 is the process of selecting a random sample, and the value of doing so. If three students are to be selected from the class for a special project, students recognize that a fair way to make the selection is to put all the student names in a box, mix them up, and draw out three names "at random." Individual students realize that they may not get selected, but that each student has the same chance of being selected. In other words, random sampling is a fair way to select a subset (a sample) of the set of interest (the population). A statistic computed from a random sample, such as the mean of the sample, can be used as an estimate of that same characteristic of the population from which the sample was selected. This estimate must be viewed with some degree of caution because of the variability in both the population and sample data. A basic tenet of statistical reasoning, then, is that random sampling allows results from a sample to be generalized to a much larger body of data, namely, the population from which the sample was selected.[7.SP.1]

"What proportion of students in the seventh grade of your school

7.SP.7a Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy.

a Develop a uniform probability model by assigning equal probability to all outcomes, and use the model to determine probabilities of events.

7.SP.7b Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy.

b Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process.

7.SP.8 Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation.

7.SP.8c Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation.

c Design and use a simulation to generate frequencies for compound events.

7.SP.1 Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.

choose football as their favorite sport?" Students realize that they do not have the time and energy to interview all seventh graders, so the next best way to get an answer is to select a random sample of seventh graders and interview them on this issue. The sample proportion is the best estimate of the population proportion, but students realize that the two are not the same and a different sample will give a slightly different estimate. In short, students realize that conclusions drawn from random samples generalize beyond the sample to the population from which the sample was selected, but a sample *statistic* is only an estimate of a corresponding population *parameter* and there will be some discrepancy between the two. Understanding variability in sampling allows the investigator to gauge the expected size of that discrepancy.

The variability in samples can be studied by means of simulation.[7.SP.2] Students are to take a random sample of 50 seventh graders from a large population of seventh graders to estimate the proportion having football as their favorite sport. Suppose, for the moment, that the true proportion is 60%, or 0.60. How much variation can be expected among the sample proportions? The scenario of selecting samples from this population can be simulated by constructing a "population" that has 60% red chips and 40% blue chips, taking a sample of 50 chips from that population, recording the number of red chips, replacing the sample in the population, and repeating the sampling process. (This can be done by hand or with the aid of technology, or by a combination of the two.) Record the proportion of red chips in each sample and plot the results.

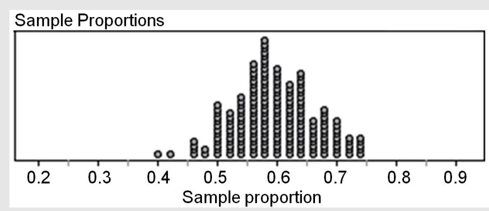
The dot plots in the margin shows results for 200 such random samples of size 50 each. Note that the sample proportions pile up around 0.60, but it is not too rare to see a sample proportion down around 0.45 or up around .0.75. Thus, we might expect a variation of close to 15 percentage points in either direction. Interestingly, about that same amount of variation persists for true proportions of 50% and 40%, as shown in the dot plots.

Students can now reason that random samples of size 50 are likely to produce sample proportions that are within about 15 percentage points of the true population value. They should now conjecture as to what will happen of the sample size is doubled or halved, and then check out the conjectures with further simulations. Why are sample sizes in public opinion polls generally around 1000 or more, rather than as small as 50?

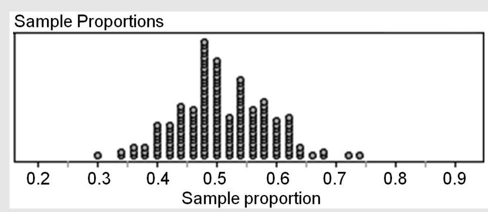
Informal comparative inference To estimate a population mean or median, the best practice is to select a random sample from that population and use the sample mean or median as the estimate, just as with proportions. But, many of the practical problems dealing with measures of center are comparative in nature, as in comparing average scores on the first and second exam or comparing average salaries between female and male employees of a firm. Such

7.SP.2 Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions.

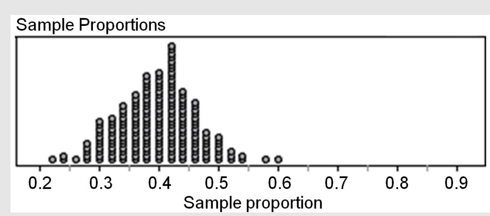
Results of simulations



Proportions of red chips in 200 random samples of size 50 from a population in which 60% of the chips are red.



Proportions of red chips in 200 random samples of size 50 from a population in which 50% of the chips are red.



Proportions of red chips in 200 random samples of size 50 from a population in which 40% of the chips are red.

comparisons may involve making conjectures about population parameters and constructing arguments based on data to support the conjectures (MP3).

If all measurements in a population are known, no sampling is necessary and data comparisons involve the calculated measures of center. Even then, students should consider variability.[7.SP.3] The figures in the margin show the female life expectancies for countries of Africa and Europe. It is clear that Europe tends to have the higher life expectancies and a much higher median, but some African countries are comparable to some of those in Europe. The mean and MAD for Africa are 53.6 and 9.5 years, respectively, whereas those for Europe are 79.3 and 2.8 years. In Africa, it would not be rare to see a country in which female life expectancy is about ten years away from the mean for the continent, but in Europe the life expectancy in most countries is within three years of the mean.

For random samples, students should understand that medians and means computed from samples will vary from sample to sample and that making informed decisions based on such sample statistics requires some knowledge of the amount of variation to expect. Just as for proportions, a good way to gain this knowledge is through simulation, beginning with a population of known structure.

The following examples are based on data compiled from nearly 200 middle school students in the Washington, DC area participating in the Census at Schools Project. Responses to the question, "How many hours per week do you usually spend on homework?," from a random sample of 10 female students and another of 10 male students from this population gave the results plotted in the margin.

Females have a slightly higher median, but students should realize that there is too much variation in the sample data to conclude that, in this population, females have a higher median homework time. An idea of how much variation to expect in samples of size 10 is needed.

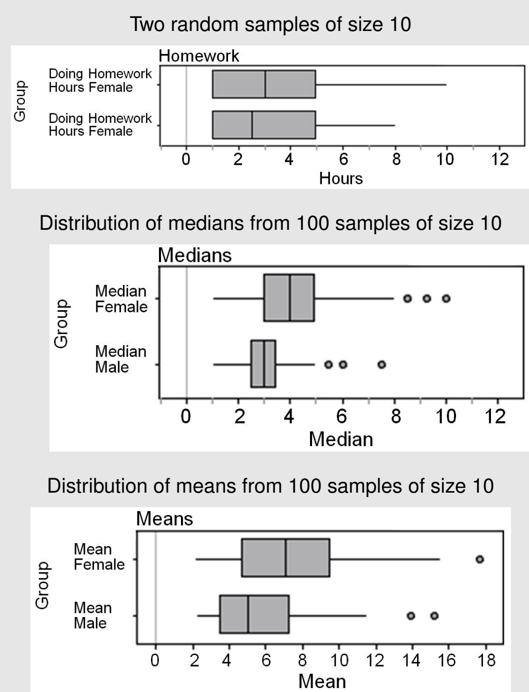
Simulation to the rescue! Students can take multiple samples of size 10 from the Census of Schools data to see how much the sample *medians* themselves tend to vary.[7.SP.4] The sample medians for 100 random samples of size 10 each, with 100 samples of males and 100 samples of females, is shown in the margin. This plot shows that the sample medians vary much less than the homework hours themselves and provides more convincing evidence that the female median homework hours is larger than that for males. Half of the female sample medians are within one hour of 4 while half of the male sample medians are within half hour of 3, although there is still overlap between the two groups.

A similar analysis based on sample means gave the results seen in the margin. Here, the overlap of the two distributions is more severe and the evidence weaker for declaring that the females have higher mean study hours than males.

[7.SP.3] Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability.

Female life expectancies in African and European countries

Hours spent on homework per week



Source: Census at Schools Project, `amstat.org/censusatschool/`

[7.SP.4] Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations.

Grade 8

Investigating patterns of association in bivariate data Students now have enough experience with coordinate geometry and linear functions[8.F.3,8.F.4,8.F.5] to plot bivariate data as points on a plane and to make use of the equation of a line in analyzing the relationship between two paired variables. They build statistical models to explore the relationship between two variables (MP4); looking for and making use of structure to describe possible association in bivariate data (MP7).

Working with paired measurement variables that might be associated linearly or in a more subtle fashion, students construct a scatter plot, describing the pattern in terms of clusters, gaps, and unusual data points (much as in the univariate situation). Then, they look for an overall positive or negative trend in the cloud of points, a linear or nonlinear (curved) pattern, and strong or weak association between the two variables, using these terms in describing the nature of the observed association between the variables.[8.SP.1]

For a data showing a linear pattern, students sketch a line through the "center" of the cloud of points that captures the essential nature of the trend, at first by use of an informal fitting procedure, perhaps as informal as laying a stick of spaghetti on the plot. How well the line "fits" the cloud of points is judged by how closely the points are packed around the line, considering that one or more outliers might have tremendous influence on the positioning of the line.[8.SP.2]

After a line is fit through the data, the slope of the line is approximated and interpreted as a rate of change, in the context of the problem.[8.F.4] The slope has important practical interpretations for most statistical investigations of this type (MP2). On the Exam 1 versus Exam 2 plot, what does the slope of 0.6 tell you about the relationship between these two sets of scores? Which students tend to do better on the second exam and which tend to do worse?[8.SP.3] Note that the negative linear trend in mammal life spans versus speed is due entirely to three long-lived, slow animals (hippo, elephant, and grizzly bear) and one short-lived, fast one (cheetah). Students with good geometry skills might explain why it would be unreasonable to expect that alligator lengths and weights would be linearly related.

Building on experience with decimals and percent, and the ideas of association between measurement variables, students now take a more careful look at possible association between categorical variables.[8.SP.4] "Is there a difference between sixth graders and eighth graders with regard to their preference for rock, rap, or country music?" Data from a random sample of sixth graders and another random sample of eighth graders are summarized by frequency counts in each cell in a two-way table of preferred music type by grade. The proportions of favored music type for the sixth graders are then compared to the proportions for eighth graders. If the two proportions for each music type are about the same, there is little or no

[8.F.3] Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear.

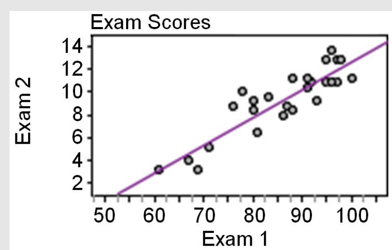
[8.F.4] Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

[8.F.5] Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

[8.SP.1] Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative association, linear association, and nonlinear association.

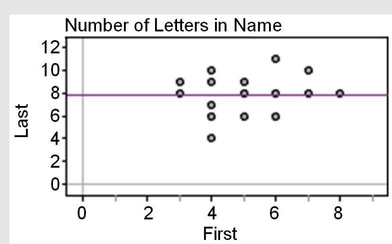
[8.SP.2] Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line.

Scores on Exam 1 and Exam 2



The least squares line fitted to the points has a positive slope and the points are closely clustered about the line, thus, the scores said to show strong positive association. Students with high scores on one exam tend to have high scores on the other. Students with low scores on one exam tend to have low scores on the other.

Letters in first and last names of students



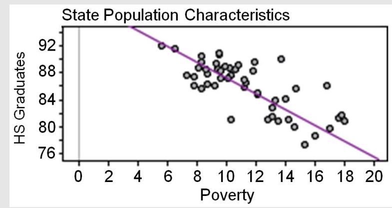
The line fitted to the points is horizontal. The number of letters in a student's first name shows no association with the number of letters in a student's last name.

[8.SP.3] Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept.

association between the grade and music preference because both grades have about the same preferences. If the two proportions differ, there is some evidence of association because grade level seems to make a difference in music preferences. The nature of the association should then be described in more detail.

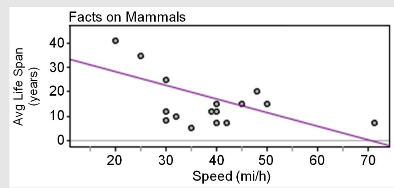
The table in the margin shows percentages of U.S. residents who have health risks due to obesity, by age category. Students should be able to explain what the cell percentages represent and provide a clear description of the nature of the association between the variables *obesity risk* and *age*. Can you tell, from this table alone, what percentage of those over the age of 18 are at risk from obesity? Such questions provide a practical mechanism for reinforcing the need for clear understanding of proportions and percentages.

High school graduation and poverty percentages for states



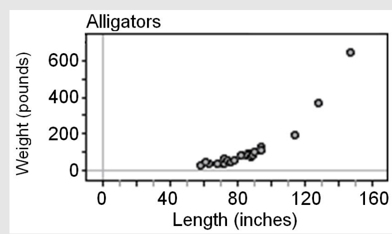
The line fitted to the data has a negative slope and data points are not all tightly clustered about the line. The percentage of a state's population in poverty shows a moderate negative association with the percentage of a state's high school graduates.

Average life span and speeds of mammals



The negative trend is due to a few outliers. This as can be seen by examining the effect of removing those points.

Weight versus length of Florida alligators



Source: http://www.factmonster.com/ipka/A0004737.html

A nonlinear association.

Table schemes for comparing frequencies and row proportions

	Rock	Rap	Country	Total
6th graders	a	b	c	d
8th graders	e	f	g	h

	Rock	Rap	Country	Total
6th graders	a/d	b/d	c/d	d
8th graders	e/h	f/h	g/h	h

Each letter represents a frequency count.

Obesity risk percentages

Age Category	Obesity		
	Not At Risk	At Risk	Row Total
Age 18 to 24	57.3	42.7	100
Age 25 to 44	38.6	61.4	100

Source: Behavioral Risk Factor Surveillance System of the Center for Disease Control

Where the Statistics and Probability Progression is heading

In high school, students build on their experience from the middle grades with data exploration and summarization, randomization as the basis of statistical inference, and simulation as a tool to understand statistical methods.

Just as Grade 6 students deepen the understanding of univariate data initially developed in elementary school, high school students deepen their understanding of bivariate data, initially developed in middle school. Strong and weak association is expressed more precisely in terms of correlation coefficients, and students become familiar with an expanded array of functions in high school that they use in modeling association between two variables.

They gain further familiarity with probability distributions generated by theory or data, and use these distributions to build an empirical understanding of the normal distribution, which is the main distribution used in measuring sampling error. For statistical methods related to the normal distribution, variation from the mean is measured by standard deviation.

Students extend their knowledge of probability, learning about conditional probability, and using probability distributions to solve problems involving expected value.

Progressions for the Common Core State Standards in Mathematics (draft)

21 April 2012

High School Statistics and Probability★

Overview

In high school, students build on knowledge and experience described in the 6-8 Statistics and Probability Progression. They develop a more formal and precise understanding of statistical inference, which requires a deeper understanding of probability. Students learn that formal inference procedures are designed for studies in which the sampling or assignment of treatments was random, and these procedures may not be informative when analyzing nonrandomized studies, often called observational studies. For example, a random selection of 100 students from your school will allow you to draw some conclusion about all the students in the school, whereas taking your class as a sample will not allow that generalization.

Probability is still viewed as long-run relative frequency but the emphasis now shifts to conditional probability and independence, and basic rules for calculating probabilities of compound events. In the plus standards• are the Multiplication Rule, probability distributions and their expected values. Probability is presented as an essential tool for decision-making in a world of uncertainty.

• Additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics is indicated by (+).

In the high school Standards, individual modeling standards are indicated by a star symbol (★). Because of its strong connection with modeling, the domain of Statistics and Probability is starred, indicating that all of its standards are modeling standards.

Interpreting categorical and quantitative data

Summarize, represent, and interpret data on a single count or measurement variable Students build on the understanding of key ideas for describing distributions—shape, center, and spread—described in the Grades 6-8 Statistics and Probability Progression. This enhanced understanding allows them to give more precise answers to deeper questions, often involving comparisons of data sets. Students use shape and the question(s) to be answered to decide on the median or mean as the more appropriate measure of center and to justify their choice through statistical reasoning. They also add a key measure of variation to their toolkits.

In connection with the mean as a measure of center, the *standard deviation* is introduced as a measure of variation. The standard deviation is based on the squared deviations from the mean, but involves much the same principle as the mean absolute deviation (MAD) that students learned about in Grades 6-8. Students should see that the standard deviation is the appropriate measure of spread for data distributions that are approximately normal in shape, as the standard deviation then has a clear interpretation related to relative frequency.

The margin shows two ways of comparing height data for males and females in the 20-29 age group. Both involve plotting the data or data summaries (box plots or histograms) on the same scale, resulting in what are called *parallel* (or *side-by-side*) *box plots* and *parallel histograms*.[S-ID.1] The parallel box plots show an obvious difference in the medians and the IQRs for the two groups; the medians for males and females are, respectively, 71 inches and 65 inches, while the IQRs are 4 inches and 5 inches. Thus, male heights center at a higher value but are slightly more variable.

The parallel histograms show the distributions of heights to be mound shaped and fairly symmetrical (approximately normal) in shape. Therefore, the data can be succinctly described using the mean and standard deviation. Heights for males and females have means of 70.4 inches and 64.7 inches, respectively, and standard deviations of 3.0 inches and 2.6 inches. Students should be able to sketch each distribution and answer questions about it just from knowledge of these three facts (shape, center, and spread). For either group, about 68% of the data values will be within one standard deviation of the mean.[S-ID.2,S-ID.3] They should also observe that the two measures of center, median and mean, tend to be close to each other for symmetric distributions.

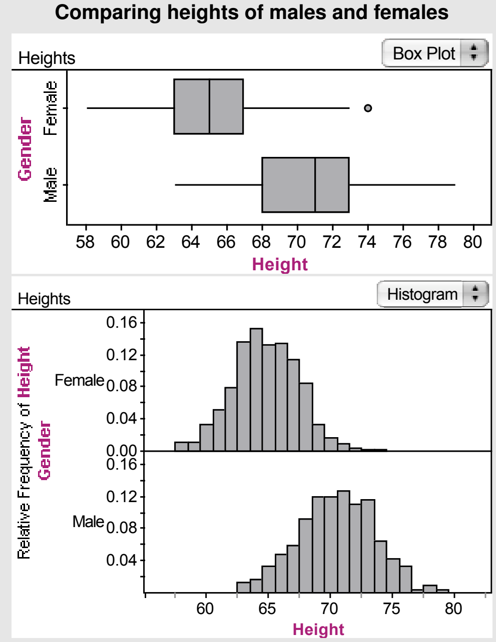
Data on heights of adults are available for anyone to look up. But how can we answer questions about standardized test scores when individual scores are not released and only a description of the distribution of scores is given? Students should now realize that we can do this only because such standardized scores generally have

Shapes of distributions

In Grade 6, students began to distinguish among distributions that were skewed or approximately symmetric. In high school, they distinguish among the latter, according to whether or not they are approximately normal, that is approximately bell-shaped.

S-ID.1 Represent data with plots on the real number line (dot plots, histograms, and box plots).

Comparing heights of males and females



Heights of U.S. males and females in the 20–29 age group. Source: *U.S. Census Bureau, Statistical Abstract of the United States: 2009*, Table 201.

S-ID.2 Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.

S-ID.3 Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).

a distribution that is mound-shaped and somewhat symmetric, i.e., approximately normal.• For example, SAT math scores for a recent year have a mean of 516 and a standard deviation of 116.• Thus, about 16% of the scores are above 632. In fact, students should be aware that technology now allows easy computation of any area under a normal curve. "If Alicia scored 680 on this SAT mathematics exam, what proportion of students taking the exam scored less than she scored?" (Answer: about 92%.)[S-ID.4]

• At this level, students are not expected to fit normal curves to data. (In fact, it is rather complicated to rescale data plots to be density plots and then find the best fitting curve.) Instead, the aim is to look for broad approximations, with application of the rather rough "empirical rule" (also called the 68%–95% Rule) for distributions that are somewhat bell-shaped. The better the bell, the better the approximation. Using such approximations is partial justification for the introduction of the standard deviation.

• See http://professionals.collegeboard.com/profdownload/2010-total-group-profile-report-cbs.pdf.

S-ID.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.

Summarize, represent, and interpret data on two categorical and quantitative variables As with univariate data analysis, students now take a deeper look at bivariate data, using their knowledge of proportions to describe categorical associations and using their knowledge of functions to fit models to quantitative data.[MP7, MP4]

MP7, MP4 Looking for patterns in tables and on scatter plots; modeling patterns in scatter plots with lines.

The table below shows statistics from the Center for Disease Control relating HIV risk to age groups. Students should be able to explain the meaning of a row or column total (marginal), a row or column percentage (conditional) or a "total" percentage (joint). They should realize that possible associations between age and HIV risk are best explained in terms of the row or column conditional percentages. Are the comparisons of percentages valid when the first age category is much smaller (in years) than the others?[S-ID.5]

S-ID.5 Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.

HIV risk by age groups, in percent of population

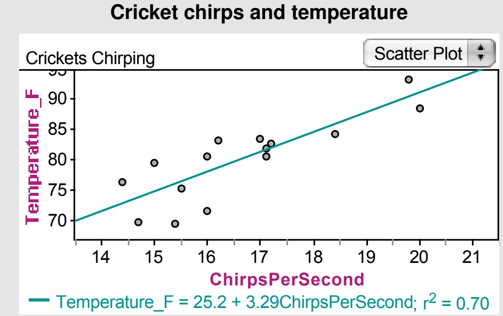
	Age	18–24	25–44	45–64	Row Total
Not at risk	Row %	14.0	59.6	26.4	100.0
	Column %	35.0	51.7	27.2	
	Total %	5.6	23.6	10.5	39.6
At risk	Row %	17.1	36.5	46.4	100.0
	Column %	65.0	48.3	72.8	
	Total %	10.3	22.0	28.1	60.4
Column total	Row %	15.9	45.6	38.5	100.0
	Column %	100.0	100.0	100.0	100.0
	Total %	15.9	45.6	38.5	100.0

Source: Center for Disease Control,
`http://apps.nccd.cdc.gov/s_broker/WEATSQL.exe/weat/freq_year.hsql`

Students have seen scatter plots in Grade 8 and now extend that knowledge to fit mathematical models that capture key elements of the relationship between two variables and to explain what the model tells us about that relationship. Some of the data should come from science, as in the examples about cricket chirps and temperature, and tree growth and age, and some from other aspects of their everyday life, e.g., cost of pizza and calories per slice (p. 6).

If you have a keen ear and some crickets, can the cricket chirps help you predict the temperature? The margin shows data modeled in a scientific investigation of that phenomenon. In this situation, the variables have been identified as chirps per second and temperature in degrees Fahrenheit. The cloud of points in the scatter plot is essentially linear with a moderately strong positive relationship. It looks like there must be something other than random behavior in

Cricket chirps and temperature



Source: George W. Pierce, *The Songs of Insects*, Harvard University Press, 1949, pp. 12–21.

this association. A model has been formulated: The least squares regression line• has been fit by technology.[S-ID.6] The model is used to draw conclusions: The line estimates that, on average, each added chirp predicts an increase of about 3.29 degrees Fahrenheit.

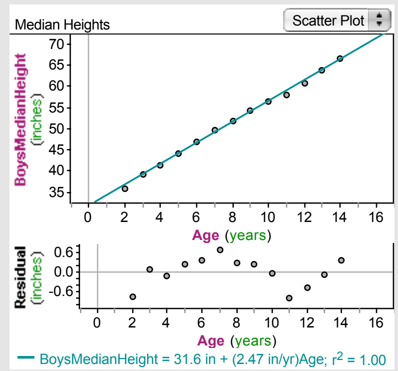
But, students must learn to take a careful look at scatter plots, as sometimes the "obvious" pattern does not tell the whole story, and can even be misleading. The margin shows the median heights of growing boys through the ages 2 to 14. The line (least squares regression line) with slope 2.47 inches per year of growth looks to be a perfect fit. [S-ID.6c] But, the *residuals*, the collection of differences between the corresponding coordinate on the least squares line and the actual data value for each age, reveal additional information. A plot of the residuals shows that growth does not proceed at a constant rate over those years.[S-ID.6b] What would be a better description of the growth pattern?

It is readily apparent to students, after a little experience with plotting bivariate data, that not all the world is linear. The figure below shows the diameters (in inches) of growing oak trees at various ages (in years). A careful look at the scatter plot reveals some curvature in the pattern,[S-ID.6a] which is more obvious in the residual plot, because the older and larger trees add to the diameter more slowly. Perhaps a curved model, such as a quadratic, will fit the data better than a line. The figure below shows that to be the case.

Would it be wise to extrapolate the quadratic model to 50-year-old trees? Perhaps a better (and simpler) model can be found by thinking in terms of cross-sectional area, rather than diameter, as the measure that might grow linearly with age.[S-ID.6a] Area is proportional to the square of the diameter, and the plot of diameter squared versus age in the margin does show remarkable linearity,[S-ID.6a] but there is always the possibility of a closer fit, that students familiar with cube root, exponential, and logarithmic functions[F-IF.7] could investigate. Students should be encouraged to think about the relationship between statistical models and the real world, and how knowledge of

• This term is used to identify the line in this Progression. Students will identify the line as the "line of best fit" obtained by technology and should not be required to use or learn "least squares regression line."

Median heights of boys



Source: National Health and Nutrition Examination Survey, 2002, `www.cdc.gov/nchs/about/major/nhanes/datatblelink.htm`.

S-ID.6 Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.

a Fit a function to the data; use functions fitted to data to solve problems in the context of the data.

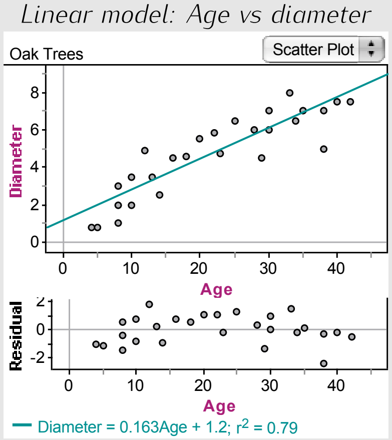
b Informally assess the fit of a function by plotting and analyzing residuals.

c Fit a linear function for a scatter plot that suggests a linear association.

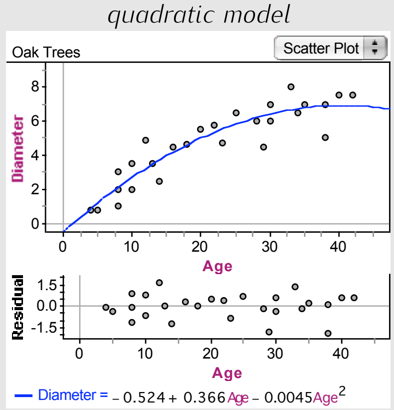
F-IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

Three iterations of the modeling cycle

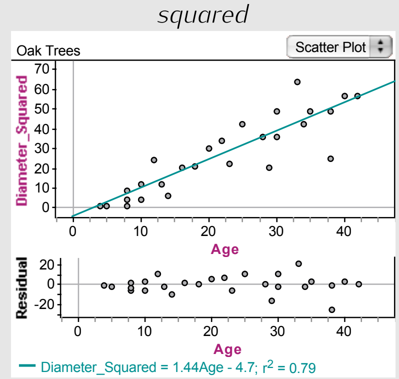
Linear model: Age vs diameter



A closer fit: Age vs diameter in a quadratic model



A simpler model: Age vs diameter squared



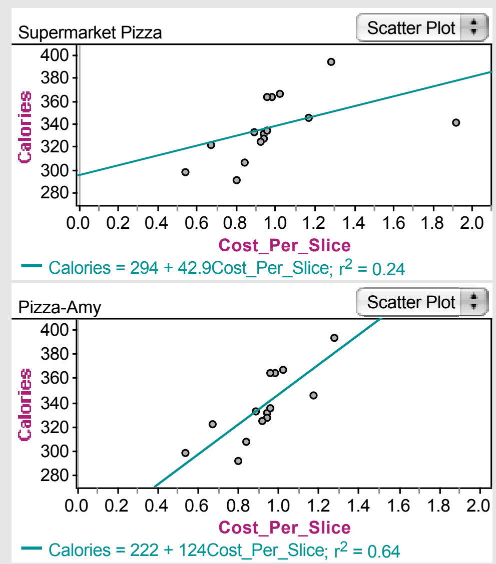
the context is essential to building good models.

Interpret linear models Students understand that the process of fitting and interpreting models for discovering possible relationships between variables requires insight, good judgment and a careful look at a variety of options consistent with the questions being asked in the investigation.[MP6]

MP6 Reasoning abstractly but quantitatively in discovering possible associations between numerical variables.

Suppose you want to see if there is a relationship between the cost per slice of supermarket pizzas and the calories per serving. The margin shows data for a sample of 15 such pizza brands, and a somewhat linear trend. A line fitted via technology might suggest that you should expect to see an increase of about 43 calories if you go from one brand to another that is one dollar more in price. But, the line does not appear to fit the data well and the correlation coefficient r (discussed below) is only about 0.5. Students will observe that there is one pizza that does not seem to fit the pattern of the others, the one with maximum cost. Why is it way out there? A check reveals that it is Amy's Organic Crust & Tomatoes, the only organic pizza in the sample. If the outlier (Amy's pizza) is removed and the discussion is narrowed to non-organic pizzas (as shown in the plot for pizzas other than Amy's), the relationship between calories and price is much stronger with an expected increase of 124 calories[S-ID.7] per extra dollar spent and a correlation coefficient of 0.8. Narrowing the question allows for a better interpretation of the slope of a line fitted to the data.[S-ID.8]

Pizza: Calories per slice vs cost



Source: *Consumer Reports*, January 2002.

S-ID.7 Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.

S-ID.8 Compute (using technology) and interpret the correlation coefficient of a linear fit.

The *correlation coefficient* measures the "tightness" of the data points about a line fitted to data, with a limiting value of 1 (or -1) if all points lie precisely on a line of positive (or negative) slope. For the line fitted to cricket chirps and temperature (p. 4), the correlation is 0.84, and for the line fitted to boys' height (p. 5), it is about 1.0. However, the quadratic model for tree growth (p. 5) is non-linear, so the value of its correlation coefficient has no direct interpretation.[S-ID.8] (The square of the correlation coefficient, however, does have an interpretation for such models.)

In situations where the correlation coefficient of a line fitted to data is close to 1 or -1, the two variables in the situation are said to have a *high correlation*. Students must see that one of the most common misinterpretations of correlation is to think of it as a synonym for causation. A high correlation between two variables (suggesting a statistical association between the two) does *not* imply that one causes the other. It is not a cost increase that causes calories to increase in pizza, and it is not a calorie increase per se that causes cost to increase; the addition of other expensive ingredients cause both to increase simultaneously.[S-ID.9] Students should look for examples of correlation being interpreted as cause and sort out why that reasoning is incorrect (MP3). Examples may include medications versus disease symptoms and teacher pay or class size versus high school graduation rates. One good way of establishing cause

S-ID.9 Distinguish between correlation and causation.

is through the design and analysis of randomized experiments, and that subject comes up in the next section.

Making inferences and justifying conclusions

Understand and evaluate random processes underlying statistical experiments Students now move beyond analyzing data to making sound statistical decisions based on probability models. The reasoning process is as follows: develop a statistical question in the form of a hypothesis (supposition) about a population parameter; choose a probability model for collecting data relevant to that parameter; collect data; compare the results seen in the data with what is expected under the hypothesis. If the observed results are far away from what is expected and have a low probability of occurring under the hypothesis, then that hypothesis is called into question. In other words, the evidence against the hypothesis is weighed by probability.[S-IC.1]

S-IC.1 Understand statistics as a process for making inferences about population parameters based on a random sample from that population.

But, what is considered "low"? That determination is left to the investigator and the circumstances surrounding the decision to be made. Statistics and probability weigh the chances; the person in charge of the investigation makes the final choice. (This is much like other areas of life in which the teacher or physician weighs the evidence and provides your chances of passing a test or easing certain disease symptoms; you make the choice.)

Consider this example. You cannot seem to roll an even number with a certain number cube. The statistical question is, "Does this number cube favor odd numbers?" The hypothesis is, "This cube does not favor odd numbers," which is the same as saying that the proportion of odd numbers rolled, in the long run, is 0.5, or the probability of tossing an odd number with this cube is 0.5. Then, toss the cube and collect data on the observed number of odds. Suppose you get an odd number in each of the:

first two tosses, which has probability $\frac{1}{4} = 0.25$ under the hypothesis;

first three tosses, which has probability $\frac{1}{8} = 0.125$ under the hypothesis;

first four tosses, which has probability $\frac{1}{16} = 0.0625$ under the hypothesis;

first five tosses, which has probability $\frac{1}{32} = 0.03125$ under the hypothesis.

At what point will students begin to seriously doubt the hypothesis that the cube does not favor odd numbers? Students should experience a number of simple situations like this to gain an understanding of how decisions based on sample data are related to probability, and that this decision process does not guarantee a correct answer to the underlying statistical question.[S-IC.3]

S-IC.3 Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.

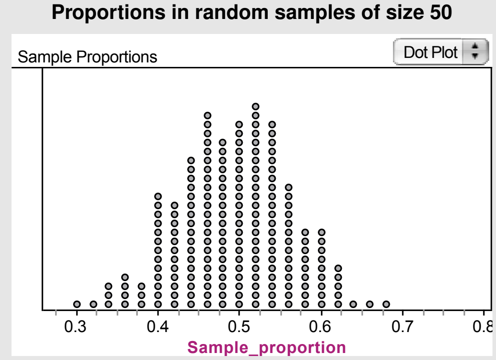
Make inferences and justify conclusions from sample surveys, experiments, and observational studies Once they see how probability intertwines with data collection and analysis, students use this knowledge to make statistical inferences from data collected in sample surveys and in designed experiments, aided by simulation and the technology that affords it.[MP5, MP3]

MP5, MP3 Using a variety of statistical tools to construct and defend logical arguments based on data.

A *Time* magazine poll reported on the status of American women. One of the statements in the poll was "It is better for a family if the father works outside the home and the mother takes care of children." Fifty-one percent of the sampled women agreed with the statement while 57% of the sampled men agreed. A note on the polling methodology states that about 1600 men and 1800 women were randomly sampled in the poll and the margin of error was about two percentage points. What is the margin of error and how is it interpreted in this context? We'll come back to the *Time* poll after exploring this question further.

"Will 50% of the homeowners in your neighborhood agree to support a proposed new tax for schools?" A student attempts to answer this question by taking a random sample of 50 homeowners in her neighborhood and asking them if they support the tax. Twenty of the sampled homeowners say they will support the proposed tax, yielding a sample proportion of $\frac{20}{50} = 0.4$. That seems like bad news for the schools, but could the population proportion favoring the tax in this neighborhood still be 50%? The student knows that a second sample of 50 homeowners might produce a different sample proportion and wonders how much variation there might be among sample proportions for samples of size 50 if, in fact, 50% is the true population proportion. Having a graphing calculator available, she simulates this sampling situation by repeatedly drawing random samples of size 50 from a population of 50% ones and 50% zeros, calculating and plotting the proportion of ones observed in each sample. The result for 200 trials is displayed in the margin. The simulated values at or below the observed 0.4 number 25 out of 200, or $\frac{25}{200} = 0.125$. So, the chance of seeing a 40% or fewer favorable response in the sample even if the true proportion of such responses was 50% is not all that small, casting little doubt on 50% as a plausible population value.

Proportions in random samples of size 50



Relating the components of this example to the statistical reasoning process, students see that the hypothesis is that the population parameter is 50% and the data are collected by a random sample. The observed sample proportion of 40% was found to be not so far from the 50% so as to cause serious doubt about the hypothesis. This lack of doubt was justified by simulating the sampling process many times and approximating the chance of a sample proportion being 40% or less under the hypothesis.[MP8]

MP8 Observing regular patterns in distributions of sample statistics.

Students now realize that there are other plausible values for the population proportion, besides 50%. The plot of the distribution of sample proportions in the margin is mound-shaped (approximately normal) and somewhat symmetric with a mean of about 0.49 (close to 0.50) and a standard deviation of about 0.07. From knowledge of the normal distribution,[S-ID.4] students knows that about 95% of the possible sample proportions that could be generated this way

S-ID.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.

will fall within two standard deviations of the mean. This two-standard deviation distance is called the *margin of error* for the sample proportions. In this example with samples of size 50, the margin of error is $2 \cdot 0.07 = 0.14$.

Suppose the true population proportion is 0.60. The distribution of the sample proportions will still look much like the plot in the margin, but the center of the distribution will be at 0.60. In this case, the observed sample proportion 0.4 will not be within the margin of error. Reasoning this way leads the student to realize that any population proportion in the interval 0.40 ± 0.14 will result in the observed sample proportion of 0.40 being within the middle 95% of the distribution of sample proportions, for samples of size 50. Thus, the interval

$$\text{observed sample proportion} \pm \text{margin of error}$$

includes the plausible values for the true population proportion in the sense that any of those populations would have produced the observed sample proportion within its middle 95% of possible outcomes. In other words, the student is confident that the proportion of homeowners in her neighborhood that will favor the tax is between 0.26 and 0.54.[S-IC.4] All of this depends on random sampling because the characteristics of distributions of sample statistics are predictable only if the sampling is random.

S-IC.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.

With regard to the *Time* poll on the status of women, the student now sees that the plausible proportions of men who agree with the statement lie between 55% and 59% while the plausible proportions of women who agree lie between 49% and 53%. What interesting conclusions might be drawn from this?[S-IC.6]

S-IC.6 Evaluate reports based on data.

Students' understanding of random sampling as the key that allows the computation of margins of error in estimating a population quantity can now be extended to the random assignment of treatments to available units in an experiment. A clinical trial in medical research, for example, may have only 50 patients available for comparing two treatments for a disease. These 50 are the population, so to speak, and randomly assigning the treatments to the patients is the "fair" way to judge possible treatment differences, just as random sampling is a fair way to select a sample for estimating a population proportion.

There is little doubt that caffeine stimulates bodily activity, but how much does it take to produce a significant effect? This is a question that involves measuring the effect of two or more treatments and deciding if the different interventions have differing effects. To obtain a partial answer to the question on caffeine, it was decided to compare a treatment consisting of 200 mg of caffeine with a control of no caffeine in an experiment involving a finger tapping exercise.

Twenty male students were randomly assigned to one of two treatment groups of 10 students each, one group receiving 200 milligrams of caffeine and the other group no caffeine. Two hours later

Finger taps per minute in a caffeine experiment

	0 mg caffeine	200 mg caffeine
	242	246
	245	248
	244	250
	248	252
	247	248
	248	250
	242	246
	244	248
	246	245
	242	250
Mean	244.8	248.3

Source: Draper and Smith, *Applied Regression Analysis*, John Wiley and Sons, 1981 the students were given a finger tapping exercise. The response is the number of taps per minute, as shown in the table.

The plot of the finger tapping data shows that the two data sets tend to be somewhat symmetric and have no extreme data points (outliers) that would have undue influence on the analysis. The sample mean for each data set, then, is a suitable measure of center, and will be used as the statistic for comparing treatments.

The mean for the 200 mg data is 3.5 taps larger than that for the 0 mg data. In light of the variation in the data, is that enough to be confident that the 200 mg treatment truly results in more tapping activity than the 0 mg treatment? In other words, could this difference of 3.5 taps be explained simply by the randomization (the luck of the draw, so to speak) rather than any real difference in the treatments? An empirical answer to this question can be found by "re-randomizing" the two groups many times and studying the distribution of differences in sample means. If the observed difference of 3.5 occurs quite frequently, then we can safely say the difference could simply be due to the randomization process. If it does not occur frequently, then we have evidence to support the conclusion that the 200 mg treatment has increased mean finger tapping count.

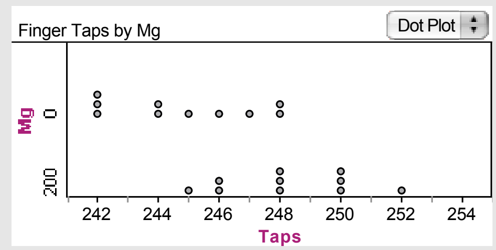
The re-randomizing can be accomplished by combining the data in the two columns, randomly splitting them into two different groups of ten, each representing 0 and 200 mg, and then calculating the difference between the sample means. This can be expedited with the use of technology.

The margin shows the differences produced in 400 re-randomizations of the data for 200 and 0 mg. The observed difference of 3.5 taps is equaled or exceeded only once out of 400 times. Because the observed difference is reproduced only 1 time in 400 trials, the data provide strong evidence that the control and the 200 mg treatment do, indeed, differ with respect to their mean finger tapping counts. In fact, we can conclude with little doubt that the caffeine is the *cause* of the increase in tapping because other possible factors should have been balanced out by the randomization.[S-IC.5] Students should be able to explain the reasoning in this decision and the nature of the error that may have been made.

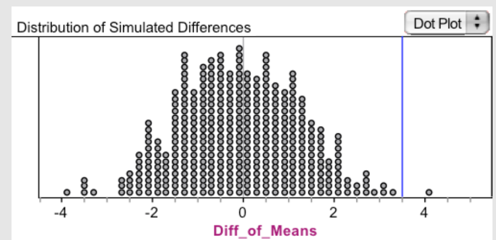
It must be emphasized repeatedly that the probabilistic reasoning underlying statistical inference is introduced into the study by way of random sampling in sample surveys and random assignment of treatments in experiments. No randomization, no such reasoning! Students will know, however, that randomization is not possible in many types of statistical investigations. Society will not condone the assigning of known harmful "treatments" (smoking, for example) to patients, so studies of the effects of smoking on health cannot be randomized experiments. Such studies must come from *observing* people who choose to smoke, as compared to those who do not, and are, therefore, called *observational studies*. The oak tree study (p. 5) and the pizza study (p. 6) are both observational studies.

Surveys of samples to estimate population parameters, random-

Plot of finger tapping data



Differences in re-randomized means for finger tapping data



S-IC.5 Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.

ized experiments to compare treatments and show cause, and observational studies to indicate possible associations among variables are the three main methods of data production in statistical studies. Students should understand the distinctions among these three and practice perceiving them in studies that are reported in the media, deciding if appropriate inferences seem to have been drawn.[S-IC.3]

S-IC.3 Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.

Conditional probability and the rules of probability

In Grades 7 and 8, students encountered the development of basic probability, including chance processes, probability models, and sample spaces. In high school, the relative frequency approach to probability is extended to conditional probability and independence, rules of probability and their use in finding probabilities of compound events, and the use of probability distributions to solve problems involving expected value. As seen in the making inferences section above, there is a strong connection between statistics and probability. This will be seen again in this section with the use of data in selecting values for probability models.

Understand independence and conditional probability and use them to interpret data In developing their understanding of conditional probability and independence, students should see two types of problems, one in which the uniform probabilities attached to outcomes leads to independence and one in which it does not. For example, suppose a student is randomly guessing the answers to all four true–false questions on a quiz. The outcomes in the sample space can be arranged as shown in the margin.[S-CP.1] Probabilities assigned to these outcomes should be equal because random guessing implies that no one outcome should be any more likely than another.

By simply counting equally likely outcomes,

$$\text{P(exactly}^{\text{MP6}}\text{ two correct answers)} = \frac{6}{16}$$

and

$$\begin{aligned}\text{P(at least one correct answer)} &= \frac{15}{16} \\ &= 1 - \text{P(no correct answers)}.\end{aligned}$$

Likewise,

$$\begin{aligned}\text{P(C on first question)} &= \frac{1}{2} \\ &= \text{P(C on second question)}\end{aligned}$$

as should seem intuitively reasonable. Now,

$$\begin{aligned}P[\text{(C on first question) and (C on second question)}] &= \frac{4}{16} \\ &= \frac{1}{4} \\ &= \frac{1}{2}\cdot\frac{1}{2},\end{aligned}$$

Possible outcomes: Guessing on four true–false questions

Number correct	Outcomes	Number correct	Outcomes	Number correct	Outcomes
4	CCCC	2	CCII	1	CIII
3	ICCC	2	CICI	1	ICII
3	CICC	2	CIIC	1	IICI
3	CCIC	2	ICCI	1	IIIC
3	CCCI	2	ICIC	0	IIII
		2	IICC		

C indicates a correct answer; I indicates an incorrect answer.

S-CP.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").

MP6 Attend to precision. "Two correct answers" may be interpreted as "at least two" or as "exactly two."

which shows that the two events (C on first question) and (C on second question) are independent, by the definition of independence.• This, too, should seem intuitively reasonable to students because the random guess on the second question should not have been influenced by the random guess on the first.

• Two events A and B are said to be independent if $P(A) \cdot P(B) = P(A \text{ and } B)$.

Students may contrast the quiz scenario above with the scenario of choosing at random two students to be leaders of a five-person working group consisting of three girls (*A*pril, *B*riana, and *C*yndi) and two boys (*D*aniel and *E*rnesto). The first name chosen indicates the discussion leader and the second the recorder, so order of selection is important. The 20 outcomes are displayed in the margin.

Selecting two students from three girls and two boys

Number of girls	Outcomes	
2	AB	BA
2	AC	CA
2	BC	CB
1	AD	DA
1	AE	EA
1	BD	DB
1	BE	EB
1	CD	DC
1	CE	EC
0	DE	ED

Here, the probability of selecting two girls is:

$$\begin{aligned} P(\text{two girls selected}) &= \frac{6}{20} \\ &= \frac{3}{10} \end{aligned}$$

whereas

$$\begin{aligned} P(\text{girl selected on first draw}) &= \frac{12}{20} \\ &= \frac{3}{5} \\ &= P(\text{girl selected on second draw}). \end{aligned}$$

Because $\frac{3}{5} \cdot \frac{3}{5} \neq \frac{3}{10}$, these two events are not independent. The selection of the second person does depend on the selection of the first when the same person cannot be selected twice.

Another way of viewing independence is to consider the conditional probability of an event A given an event B, $P(A|B)$, as the probability of A in the sample space restricted to just those outcomes that constitute B. In the table of outcomes for guessing on the true-false questions,

$$\begin{aligned} P(\text{C on second question} \mid \text{C on first question}) &= \frac{4}{8} \\ &= \frac{1}{2} \\ &= P(\text{C on second}) \end{aligned}$$

and students see that knowledge of what happened on the first question does not alter the probability of the outcome on the second; the two events are independent.

In the selecting students scenario, the conditional probability of a girl on the second selection, given that a girl was selected on the first is

$$\begin{aligned} P(\text{girl on second} \mid \text{girl on first}) &= \frac{6}{12} \\ &= \frac{1}{2} \end{aligned}$$

and

$$\text{P(girl on second)} = \frac{3}{5}.$$

So, these two events are again seen to be dependent. The outcome of the second draw does depend on what happened at the first draw.[S-CP.3]

S-CP.3 Understand the conditional probability of A given B as $P(A \text{ and } B)/P(B)$, and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B.

Students understand that in real world applications the probabilities of events are often approximated by data about those events. For example, the percentages in the table for HIV risk by age group (p. 4) can be used to approximate probabilities of HIV risk with respect to age or age with respect to HIV risk for a randomly selected adult from the U.S. population of adults. Emphasizing the conditional nature of the row and column percentages:

$$\text{P(adult is age 18 to 24 | adult is at risk)} = 0.171$$

whereas

$$\text{P(adult is at risk | adult is age 18 to 24)} = 0.650.$$

Comparing the latter to

$$\text{P(adult is at risk | adult is age 25 to 44)} = 0.483$$

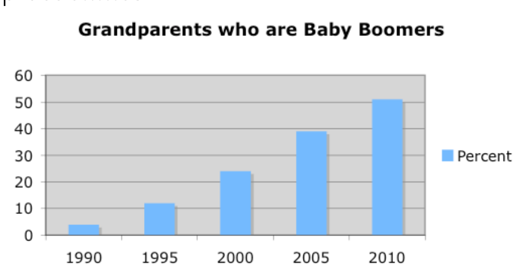
shows that the conditional distributions change from column to column, reflecting dependence and an association between age category and HIV risk.[S-CP.4, S-CP.5]

S-CP.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities.

S-CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.

Students can gain practice in interpreting percentages and using them as approximate probabilities from study data presented in the popular press. Quite often the presentations are a little confusing and can be interpreted in more than one way. For example, two data summaries from *USA Today* are shown below. What might these percentages represent and how might they be used as approximate probabilities?[S-CP.5]

S-CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.



Top age groups for DUI	
21–25	29%
26–29	24%
18–20	20%
30–34	19%

Use the rules of probability to compute probabilities of compound events in a uniform probability model The two-way table for HIV risk by age group (p. 4) gives percentages from a data analysis that can be used to approximate probabilities, but students realize that such tables can be developed from theoretical probability models. Suppose, for example, two fair six-sided number cubes are rolled, giving rise to 36 equally likely outcomes.

Outcomes for specified events can be diagramed as sections of the table, and probabilities calculated by simply counting outcomes. This type of example is one way to review information on conditional probability and introduce the addition and multiplication rules. For example, defining events:

A is "you roll numbers summing to 8 or more"

B is "you roll doubles"

and counting outcomes leads to

$$\begin{aligned} P(A) &= \frac{15}{36} \\ P(B) &= \frac{6}{36} \\ P(A \text{ and } B) &= \frac{3}{36}, \quad \text{and} \\ P(B|A) &= \frac{3}{15}, \quad \text{the fraction of A's 15 outcomes that also fall in B.}^{\text{S-CP.6}} \end{aligned}$$

Now, by counting outcomes

$$P(A \text{ or } B) = \frac{18}{36}$$

or by using the Addition Rule[S-CP.7]

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ &= \frac{15}{36} + \frac{6}{36} - \frac{3}{36} \\ &= \frac{18}{36}. \end{aligned}$$

\+ By the Multiplication Rule[S-CP.8]

$$\begin{aligned} P(A \text{ and } B) &= P(A)P(B|A) \\ &= \frac{15}{36} \cdot \frac{3}{15} \\ &= \frac{3}{36}. \end{aligned}$$

The assumption that all outcomes of rolling each cube once are equally likely results in the outcome of rolling one cube being independent of the outcome of rolling the other.[S-CP.5] Students should understand that independence is often used as a simplifying assumption in constructing theoretical probability models that approximate real situations. Suppose a school laboratory has two smoke alarms as a built in redundancy for safety. One has probability 0.4 of going off when steam (not smoke) is produced by running hot water and the other has probability 0.3 for the same event. The probability

Possible outcomes: Rolling two number cubes

	1	2	3	4	5	6
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

S-CP.6 Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model.

S-CP.7 Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model.

S-CP.8 (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interpret the answer in terms of the model.

S-CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.

that they both go off the next time someone runs hot water in the sink can be reasonably approximated as the product $0.4 \cdot 0.3 = 0.12$, even though there may be some dependence between two systems operating in the same room. Modeling independence is much easier than modeling dependence, but models that assume independence are still quite useful.

Using probability to make decisions

Calculate expected values and use them to solve problems As students gain experience with probability problems that deal with listing and counting outcomes, they will come to realize that, most often, applied problems concern some numerical quantity of interest rather than a description of the outcomes themselves.MP1 MP2 Advertisers want to know how many customers will purchased their product, not the order in which they came into the store. A political pollster wants to know how many people are likely to vote for a particular candidate and a student wants to know how many questions he is likely to get right by guessing on a true-false quiz.

In such situations, the outcomes can be seen as numerical values of a *random variable*.• Reconfiguring the tables of outcomes for the true-false test (p. 13) and student selection (p. 14) in a way that emphasizes these numerical values and their probabilities gives rise to the probability distributions shown below.

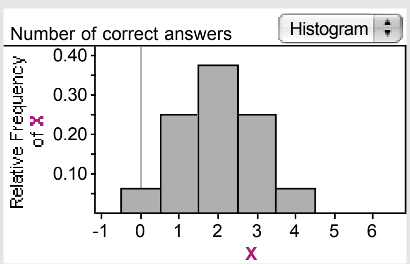
MP1 Make sense of a problem, analyzing givens, constraints, relationships, and goals.

MP2 Formulate a probability model for a practical problem that reflects constraints and relationships, and reason abstractly to solve the problem.

• Students should realize that random variables are different from the variables used in other high school domains; random variables are functions of the outcomes of a random process and thus have probabilities attached to their possible values.

True–false test

Number correct, X	Probability
0	$\frac{1}{16}$
1	$\frac{4}{16}$
2	$\frac{6}{16}$
3	$\frac{4}{16}$
4	$\frac{1}{16}$



Selection of students

Number of girls, Y	Probability
0	$\frac{1}{10}$
1	$\frac{6}{10}$
2	$\frac{3}{10}$

Because probability is viewed as a long-run relative frequency, probability distributions can be treated as theoretical data distributions. If 1600 students all guessed at all four questions on the true-false test, about 400 of them would get three answers correct, about 100 four answers correct, and so on. These scores could then be averaged to come up with a mean score of:

$$0 \cdot \frac{1}{16} + 1 \cdot \frac{4}{16} + 2 \cdot \frac{6}{16} + 3 \cdot \frac{4}{16} + 4 \cdot \frac{1}{16} = 2.$$

With the *number correct* labeled as X, this value is called the *expected value* of X, usually expressed as E(X). Anyone guessing at all four true-false questions on a test can expect, over the long run, to get two correct answers per test, which is intuitively reasonable.

Students then develop the general rule that, for any discrete• random variable X,

$$E(X) = \sum (\text{value of X})(\text{probability of that value})$$

where the sum extends over all values of X.S-MD.2

• Students need not learn the term "discrete random variable." All of the random variables treated in this Progression are discrete random variables, that is, they concern only sample spaces which are collections of discrete objects.

S-MD.2 (+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution.

For the random variable *number of girls*, Y, E(Y) = 1.2. Of course, 1.2 girls cannot be selected in any one group, but if the group selects leaders at random each day for ten days, they would be expected to choose about 12 girls as compared to 8 boys over the period.

The probability distributions considered above arise from theoretical probability models, but they can also come from empirical approximations. The margin displays the distribution of family sizes in the U.S., according to the Census Bureau. (Very few families have more than seven members.) These proportions calculated from census counts can serve as to approximate probabilities that families of given sizes will be selected in a random sample. If an advertiser randomly samples 1000 families for a special trial of a new product to be used by all members of the family, she would expect to have the product used by about 3.49 people per family, or about 3,490 people over all.

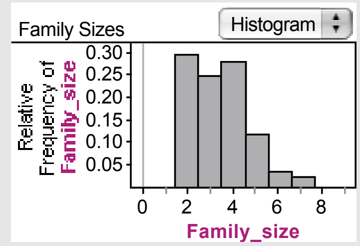
Use probability to evaluate outcomes of decisions Students should understand that probabilities and expected values must be thought of as long-term relative frequencies and means, and consider the implications of that view in decision making. Consider the following real-life example. The Wisconsin lottery had a game called "Hot Potato" that cost a dollar to play and had payoff probabilities as shown in the margin. The sum of these probabilities is not 1, but there is a key payoff value missing from the table. Students can include that key value and its probability to make this a true probability distribution and find that the expected payoff per game is about \$0.55.[S-MD.5] Losing a dollar to play the game may not mean much to an individual player, but expecting to take in \$450 for every \$1000 spent on the game means a great deal to the Wisconsin Lottery Commission!

Studying the behavior of games of chance is fun, but students must see more serious examples such as this one, based on empirical data. In screening for HIV by use of both the ELISA and Western Blot tests, HIV-positive males will test positive in 99.9% of the cases and HIV-negative males will test negative in 99.99% of the cases. Among men with low-risk behavior, the rate of HIV is about 1 in 10,000. What is the probability that a low-risk male who tests positive actually is HIV positive?

Having students turn the given rates into expected counts and placing the counts in an appropriate table is a good way for them to construct a meaningful picture of what is going on here. There are two variables, whether or not a tested person is HIV positive and whether or not the test is positive. Starting with a cohort of 10,000 low-risk males, the table might look like the one in the margin. The conditional probability of a randomly selected male being HIV positive, given that he tested positive is about 0.5! Students should discuss the implications of this in relation to decisions concerning mass screening for HIV.[S-MD.6, S-MD.7]

Empirical distribution of family size

Family size	Proportion
2	0.28
3	0.25
4	0.27
5	0.13
6	0.04
7	0.03



Source: U.S. Census Bureau, `http://www.census.gov/population/www/socdemo/hh-fam/cps2010.html`; Table F1

"Hot Potato" payoffs and probabilities

Payoff (\$)	Probability
1	$\frac{1}{9}$
2	$\frac{1}{13}$
3	$\frac{1}{43}$
6	$\frac{1}{94}$
9	$\frac{1}{150}$
18	$\frac{1}{300}$
50	$\frac{1}{2050}$
100	$\frac{1}{144000}$
300	$\frac{1}{180000}$
900	$\frac{1}{270000}$

For details about Hot Potato and other lotteries, see `www.wilottery.com/scratchgames/historical.aspx`.

S-MD.5 (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

HIV testing expected frequencies

	HIV+ male	HIV- male	Totals
HIV+ test result	0.999	1	1.999
HIV- test result	0.001	9,998	9,998.001
Totals	1	9,999	10,000

S-MD.6 (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

S-MD.7 (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).

Where the Statistics and Probability Progression might lead

Careers A few examples of careers that draw on the knowledge discussed in this Progression are actuary, manufacturing technician, industrial engineer or statistician, industrial engineer and production manager. The level of education required for these careers and sources of further information and examples of workplace tasks are summarized in the table below. Information about careers for statisticians in health and medicine, business and industry, and government appears on the web site of the American Statistical Association (`www.amstat.org/careers/index.cfm`).

	Education	Location of information, workplace task
Actuary	bachelors	*Ready or Not*, p. 79; `http://beanactuary.org/how/highschool/`
Manufacturing technician	associate	*Ready or Not*, p. 81
Industrial engineer or statistician	bachelors	`http://www.achieve.org/node/205`
Industrial engineer; production manager	bachelors	`http://www.achieve.org/node/620`

Source: *Ready or Not: Creating a High School Diploma That Counts*, 2004, `www.achieve.org/ReadyorNot`

College Most college majors in the sciences (including health sciences), social sciences, biological sciences (including agriculture), business, and engineering require some knowledge of statistics. Typically, this exposure begins with a non-calculus-based introductory course that would expand the empirical view of statistical inference found in this high school progression to a more general view based on mathematical formulations of inference procedures. (The Advanced Placement Statistics course is at this level.) After that general introduction, those in more applied areas would take courses in statistical modeling (regression analysis) and the design and analysis of experiments and/or sample surveys. Those heading to degrees in mathematics, statistics, economics, and more mathematical areas of engineering would study the mathematical theory of statistics and probability at a deeper level, perhaps along with more specialized courses in, say, time series analysis or categorical data analysis. Whatever their future holds, most students will encounter data in their chosen field—and lots of it. So, gaining some knowledge of both applied and theoretical statistics, along with basic skills in computing, will be a most valuable asset indeed!

Progressions for the Common Core State Standards in Mathematics (draft)

21 April 2012

K–5, Number and Operations in Base Ten

Overview

Students' work in the base-ten system is intertwined with their work on counting and cardinality, and with the meanings and properties of addition, subtraction, multiplication, and division. Work in the base-ten system relies on these meanings and properties, but also contributes to deepening students' understanding of them.

Position The base-ten system is a remarkably efficient and uniform system for systematically representing all numbers. Using only the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, every number can be represented as a string of digits, where each digit represents a value that depends on its place in the string. The relationship between values represented by the places in the base-ten system is the same for whole numbers and decimals: the value represented by each place is always 10 times the value represented by the place to its immediate right. In other words, moving one place to the left, the value of the place is multiplied by 10. In moving one place to the right, the value of the place is divided by 10. Because of this uniformity, standard algorithms for computations within the base-ten system for whole numbers extend to decimals.

Base-ten units Each place of a base-ten numeral represents a base-ten unit: ones, tens, tenths, hundreds, hundredths, etc. The digit in the place represents 0 to 9 of those units. Because ten like units make a unit of the next highest value, only ten digits are needed to represent any quantity in base ten. The basic unit is a *one* (represented by the rightmost place for whole numbers). In learning about whole numbers, children learn that ten ones compose a new kind of unit called a *ten*. They understand two-digit numbers as composed of tens and ones, and use this understanding in computations, decomposing 1 ten into 10 ones and composing a ten from 10 ones.

The power of the base-ten system is in repeated bundling by ten: 10 tens make a unit called a hundred. Repeating this process of creating new units by bundling in groups of ten creates units called *thousand*, *ten thousand*, *hundred thousand* .... In learning about decimals, children partition a one into 10 equal-sized smaller units, each of which is a tenth. Each base-ten unit can be understood in terms of any other base-ten unit. For example, one hundred can be viewed as a tenth of a thousand, 10 tens, 100 ones, or 1,000 tenths. Algorithms for operations in base ten draw on such relationships among the base-ten units.

Computations Standard algorithms for base-ten computations with the four operations rely on decomposing numbers written in base-ten notation into base-ten units. The properties of operations then allow any multi-digit computation to be reduced to a collection of single-digit computations. These single-digit computations sometimes require the composition or decomposition of a base-ten unit.

Beginning in Kindergarten, the requisite abilities develop gradually over the grades. Experience with addition and subtraction within 20 is a Grade 1 standard[1.OA.6] and fluency is a Grade 2 standard.[2.OA.2] Computations within 20 that "cross 10," such as $9+8$ or $13-6$, are especially relevant to NBT because they afford the development of the Level 3 make-a-ten strategies for addition and subtraction described in the OA Progression. From the NBT perspective, make-a-ten strategies are (implicitly) the first instances of composing or decomposing a base-ten unit. Such strategies are a foundation for understanding in Grade 1 that addition may require composing a ten[1.NBT.4] and in Grade 2 that subtraction may involve decomposing a ten.[2.NBT.7]

Strategies and algorithms The Standards distinguish strategies from algorithms.• For example, students use strategies for addition and subtraction in Grades K-3, but are expected to fluently add and subtract whole numbers using standard algorithms by the end of Grade 4. Use of the standard algorithms can be viewed as the culmination of a long progression of reasoning about quantities, the base-ten system, and the properties of operations.

This progression distinguishes between two types of computational strategies: special strategies and general methods. For example, a special strategy for computing $398+17$ is to decompose 17 as $2+15$, and evaluate $(398+2)+15$. Special strategies either cannot be extended to all numbers represented in the base-ten system or require considerable modification in order to do so. A more readily generalizable method of computing $398+17$ is to combine like base-ten units. General methods extend to all numbers represented in the base-ten system. A general method is not necessarily efficient. For example, counting on by ones is a general method that can be easily modified for use with finite decimals. General methods based on place value, however, are more efficient and can be viewed as closely connected with standard algorithms.

1.OA.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8+6=8+2+4=10+4=14$); decomposing a number leading to a ten (e.g., $13-4=13-3-1=10-1=9$); using the relationship between addition and subtraction (e.g., knowing that $8+4=12$, one knows $12-8=4$); and creating equivalent but easier or known sums (e.g., adding $6+7$ by creating the known equivalent $6+6+1=12+1=13$).

2.OA.2 Fluently add and subtract within 20 using mental strategies.[1] By end of Grade 2, know from memory all sums of two one-digit numbers.

1.NBT.4 Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.

2.NBT.7 Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

• **Computation algorithm.** A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. See also: computation strategy.

Computation strategy. Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. See also: computation algorithm.

Mathematical practices Both general methods and special strategies are opportunities to develop competencies relevant to the NBT standards. Use and discussion of both types of strategies offer opportunities for developing fluency with place value and properties of operations, and to use these in justifying the correctness of computations (MP.3). Special strategies may be advantageous in situations that require quick computation, but less so when uniformity is useful. Thus, they offer opportunities to raise the topic of using appropriate tools strategically (MP.5). Standard algorithms can be viewed as expressions of regularity in repeated reasoning (MP.8) used in general methods based on place value.

Numerical expressions and recordings of computations, whether with strategies or standard algorithms, afford opportunities for students to contextualize, probing into the referents for the symbols involved (MP.2). Representations such as bundled objects or math drawings (e.g., drawings of hundreds, tens, and ones) and diagrams (e.g., simplified renderings of arrays or area models) afford the mathematical practice of explaining correspondences among different representations (MP.1). Drawings, diagrams, and numerical recordings may raise questions related to precision (MP.6), e.g., does that 1 represent 1 one or 1 ten? This progression gives examples of representations that can be used to connect numerals with quantities and to connect numerical representations with combination, composition, and decomposition of base-ten units as students work towards computational fluency.

Kindergarten

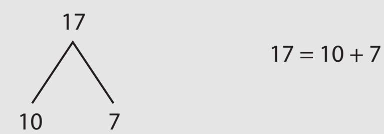
In Kindergarten, teachers help children lay the foundation for understanding the base-ten system by drawing special attention to 10. Children learn to view the whole numbers 11 through 19 as ten ones and some more ones. They decompose 10 into pairs such as $1+9$, $2+8$, $3+7$ and find the number that makes 10 when added to a given number such as 3 (see the OA Progression for further discussion).

Work with numbers from 11 to 19 to gain foundations for place value[K.NBT.1] Children use objects, math drawings,• and equations to describe, explore, and explain how the "teen numbers," the counting numbers from 11 through 19, are ten ones and some more ones. Children can count out a given teen number of objects, e.g., 12, and group the objects to see the ten ones and the two ones. It is also helpful to structure the ten ones into patterns that can be seen as ten objects, such as two fives (see the OA Progression).

K.NBT.1 Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

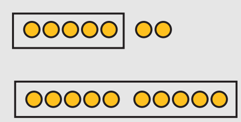
• Math drawings are simple drawings that make essential mathematical features and relationships salient while suppressing details that are not relevant to the mathematical ideas.

Number-bond diagram and equation



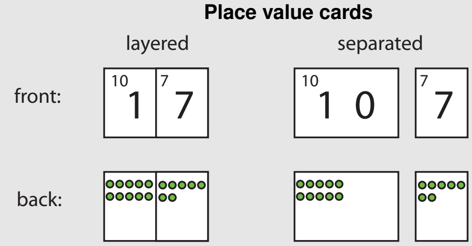
Decompositions of teen numbers can be recorded with diagrams or equations.

5- and 10-frames



Children can place small objects into 10-frames to show the ten as two rows of five and the extra ones within the next 10-frame, or work with strips that show ten ones in a column.

Place value cards



Children can use layered place value cards to see the 10 "hiding" inside any teen number. Such decompositions can be connected to numbers represented with objects and math drawings.

A difficulty in the English-speaking world is that the words for teen numbers do not make their base-ten meanings evident. For example, "eleven" and "twelve" do not sound like "ten and one" and "ten and two." The numbers "thirteen, fourteen, fifteen, ..., nineteen" reverse the order of the ones and tens digits by saying the ones digit first. Also, "teen" must be interpreted as meaning "ten" and the prefixes "thir" and "fif" do not clearly say "three" and "five." In contrast, the corresponding East Asian number words are "ten one, ten two, ten three," and so on, fitting directly with the base-ten structure and drawing attention to the role of ten. Children could learn to say numbers in this East Asian way in addition to learning the standard English number names. Difficulties with number words beyond nineteen are discussed in the Grade 1 section.

The numerals $11, 12, 13, \ldots, 19$ need special attention for children to understand them. The first nine numerals $1, 2, 3, \ldots, 9$, and 0 are essentially arbitrary marks. These same marks are used again to represent larger numbers. Children need to learn the differences in the ways these marks are used. For example, initially, a numeral such as 16 looks like "one, six," not "1 ten and 6 ones ." Layered place value cards can help children see the 0 "hiding" under the ones place and that the 1 in the tens place really is 10 (ten ones).

By working with teen numbers in this way in Kindergarten, students gain a foundation for viewing 10 ones as a new unit called a ten in Grade 1.

Grade 1

In first grade, students learn to view ten ones as a unit called a ten. The ability to compose and decompose this unit flexibly and to view the numbers 11 to 19 as composed of one ten and some ones allows development of efficient, general base-ten methods for addition and subtraction. Students see a two-digit numeral as representing some tens and they add and subtract using this understanding.

Extend the counting sequence and understand place value Through practice and structured learning time, students learn patterns in spoken number words and in written numerals, and how the two are related.

Grade 1 students take the important step of viewing ten ones as a unit called a "ten."[1.NBT.2a] They learn to view the numbers 11 through 19 as composed of 1 ten and some ones.[1.NBT.2b] They learn to view the decade numbers $10, \ldots, 90$, in written and in spoken form, as 1 ten, ..., 9 tens.[1.NBT.2c] More generally, first graders learn that the two digits of a two-digit number represent amounts of tens and ones, e.g., 67 represents 6 tens and 7 ones.

1.NBT.2 Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:

- a 10 can be thought of as a bundle of ten ones—called a "ten."
- b The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.
- c The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).

The number words continue to require attention at first grade because of their irregularities. The decade words, "twenty," "thirty," "forty," etc., must be understood as indicating 2 tens, 3 tens, 4 tens, etc. Many decade number words sound much like teen number words. For example, "fourteen" and "forty" sound very similar, as do "fifteen" and "fifty," and so on to "nineteen" and "ninety." As discussed in the Kindergarten section, the number words from 13 to 19 give the number of ones before the number of tens. From 20 to 100, the number words switch to agreement with written numerals by giving the number of tens first. Because the decade words do not clearly indicate they mean a number of tens ("-ty" does mean tens but not clearly so) and because the number words "eleven" and "twelve" do not cue students that they mean "1 ten and 1" and "1 ten and 2," children frequently make count errors such as "twenty-nine, twenty-ten, twenty-eleven, twenty-twelve."

Grade 1 students use their base-ten work to help them recognize that the digit in the tens place is more important for determining the size of a two-digit number.[1.NBT.3] They use this understanding to compare two two-digit numbers, indicating the result with the symbols $>$, $=$, and $<$. Correctly placing the $<$ and $>$ symbols is a challenge for early learners. Accuracy can improve if students think of putting the wide part of the symbol next to the larger number.

1.NBT.3 Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols $>$, $=$, and $<$.

Use place value understanding and properties of operations to add and subtract First graders use their base-ten work to compute sums within 100 with understanding.[1.NBT.4] Concrete objects, cards, or drawings afford connections with written numerical work and discussions and explanations in terms of tens and ones. In particular, showing composition of a ten with objects or drawings

1.NBT.4 Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.

affords connection of the visual ten with the written numeral 1 that indicates 1 ten.

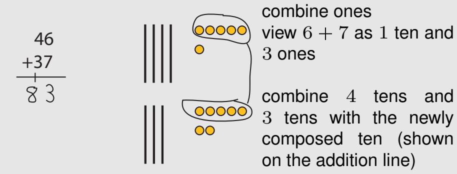
Adding tens and ones separately as illustrated in the margin is a general method that can extend to any sum of multi-digit numbers. Students may also develop sequence methods that extend their Level 2 single-digit counting on strategies (see the OA Progression) to counting on by tens and ones, or mixtures of such strategies in which they add instead of count the tens or ones. Using objects or drawings of 5-groups can support students' extension of the Level 3 make-a-ten methods discussed in the OA Progression for single-digit numbers.

First graders also engage in mental calculation, such as mentally finding 10 more or 10 less than a given two-digit number without having to count by ones.[1.NBT.5] They may explain their reasoning by saying that they have one more or one less ten than before. Drawings and layered cards can afford connections with place value and be used in explanations.

In Grade 1, children learn to compute differences of two-digit numbers for limited cases.[1.NBT.6] Differences of multiples of 10, such as $70 - 40$ can be viewed as 7 tens minus 4 tens and represented with concrete models such as objects bundled in tens or drawings. Children use the relationship between subtraction and addition when they view $80 - 70$ as an unknown addend addition problem, $70 + \square = 80$, and reason that 1 ten must be added to 70 to make 80, so $80 - 70 = 10$.

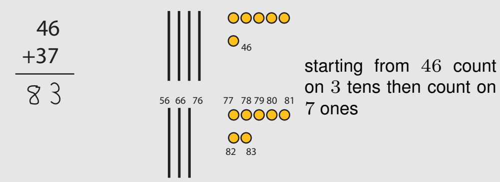
First graders are not expected to compute differences of two-digit numbers other than multiples of ten. Deferring such work until Grade 2 allows two-digit subtraction with and without decomposing to occur in close succession, highlighting the similarity between these two cases.

General method: Adding tens and ones separately



This method is an application of the associative property.

Special strategy: Counting on by tens



This strategy requires counting on by tens from 46, beginning 56, 66, 76, then counting on by ones.

1.NBT.5 Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used.

1.NBT.6 Subtract multiples of 10 in the range 10–90 from multiples of 10 in the range 10–90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Grade 2

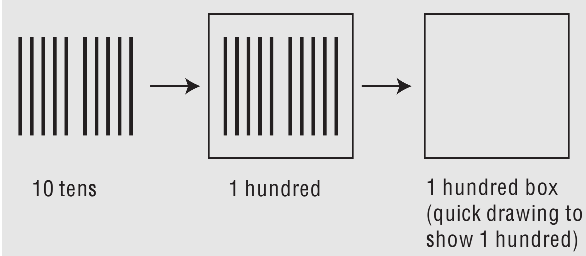
At Grade 2, students extend their base-ten understanding to hundreds. They now add and subtract within 1000, with composing and decomposing, and they understand and explain the reasoning of the processes they use. They become fluent with addition and subtraction within 100.

Understand place value In Grade 2, students extend their understanding of the base-ten system by viewing 10 tens as forming a new unit called a "hundred."[2.NBT.1a] This lays the groundwork for understanding the structure of the base-ten system as based in repeated bundling in groups of 10 and understanding that the unit associated with each place is 10 of the unit associated with the place to its right.

2.NBT.1a Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:

a 100 can be thought of as a bundle of ten tens—called a "hundred."

Drawings to support seeing 10 tens as 1 hundred



Representations such as manipulative materials, math drawings and layered three-digit place value cards afford connections between written three-digit numbers and hundreds, tens, and ones. Number words and numbers written in base-ten numerals and as sums of their base-ten units can be connected with representations in drawings and place value cards, and by saying numbers aloud and in terms of their base-ten units, e.g., 456 is "Four hundred fifty six" and "four hundreds five tens six ones."[2.NBT.3]

2.NBT.3 Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.

Unlike the decade words, the hundred words indicate base-ten units. For example, it takes interpretation to understand that "fifty" means five tens, but "five hundred" means almost what it says ("five hundred" rather than "five hundreds"). Even so, this doesn't mean that students automatically understand 500 as 5 hundreds; they may still only think of it as the number reached after 500 counts of 1.

Students begin to work towards multiplication when they skip count by 5s, by 10s, and by 100s. This skip counting is not yet true multiplication because students don't keep track of the number of groups they have counted.[2.NBT.2]

2.NBT.2 Count within 1000; skip-count by 5s, 10s, and 100s.

Comparing magnitudes of two-digit numbers draws on the understanding that 1 ten is greater than any amount of ones represented by a one-digit number. Comparing magnitudes of three-digit numbers draws on the understanding that 1 hundred (the smallest three-digit number) is greater than any amount of tens and ones represented by a two-digit number. For this reason, three-digit numbers are compared by first inspecting the hundreds place (e.g. $845 > 799$; $849 < 855$).[2.NBT.4]

2.NBT.4 Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using $>$, =, and $<$ symbols to record the results of comparisons.

Use place value understanding and properties of operations to add and subtract Students become fluent in two-digit addition and subtraction.[2.NBT.5, 2.NBT.6] Representations such as manipulative materials and drawings may be used to support reasoning and explanations about addition and subtraction with three-digit numbers.[2.NBT.7] When students add ones to ones, tens to tens, and hundreds to hundreds they are implicitly using a general method based on place

2.NBT.5 Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

2.NBT.6 Add up to four two-digit numbers using strategies based on place value and properties of operations.

2.NBT.7 Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

value and the associative and commutative properties of addition. Examples of how general methods can be represented in numerical work and composition and decomposition can be represented in math drawings are shown in the margin.

Drawings and diagrams can illustrate the reasoning repeated in general methods for computation that are based on place value. These provide an opportunity for students to observe this regularity and build toward understanding the standard addition and subtraction algorithms required in Grade 4 as expressions of repeated reasoning (MP.8).

At Grade 2, composing and decomposing involves an extra layer of complexity beyond that of Grade 1. This complexity manifests itself in two ways. First, students must understand that a hundred is a unit composed of 100 ones, but also that it is composed of 10 tens. Second, there is the possibility that both a ten and a hundred are composed or decomposed. For example, in computing $398+7$ a new ten and a new hundred are composed. In computing $302-184$, a ten and a hundred are decomposed.

Students may continue to develop and use special strategies for particular numerical cases or particular problem situations such as Unknown Addend. For example, instead of using a general method to add $398+7$, students could reason mentally by decomposing the 7 ones as $2+5$, adding 2 ones to 398 to make 400, then adding the remaining 5 ones to make 405. This method uses the associative property of addition and extends the make-a-ten strategy described in the OA Progression. Or students could reason that 398 is close to 400, so the sum is close to $400+7$, which is 407, but this must be 2 too much because 400 is 2 more than 398, so the actual sum is 2 less than 407, which is 405. Both of these strategies make use of place value understanding and are practical in limited cases.

Subtractions such as $302-184$ can be computed using a general method by decomposing a hundred into 10 tens, then decomposing one of those tens into 10 ones. Students could also view it as an unknown addend problem $184+\square=302$, thus drawing on the relationship between subtraction and addition. With this view, students can solve the problem by adding on to 184: first add 6 to make 190, then add 10 to make 200, next add 100 to make 300, and finally add 2 to make 302. They can then combine what they added on to find the answer to the subtraction problem: $6+10+100+2=118$. This strategy is especially useful in unknown addend situations. It can be carried out more easily in writing because one does not have to keep track of everything mentally. This is a Level 3 strategy, and is easier than the Level 3 strategy illustrated below that requires keeping track of how much of the second addend has been added on. (See the OA Progression for further discussion of levels.)

When computing sums of three-digit numbers, students might use strategies based on a flexible combination of Level 3 composition and decomposition and Level 2 counting-on strategies when finding the value of an expression such as $148+473$. For exam-

Addition: Recording combined hundreds, tens, and ones on separate lines

456 +167	456 +167 500	456 +167 500 110	456 +167 500 110 13 623

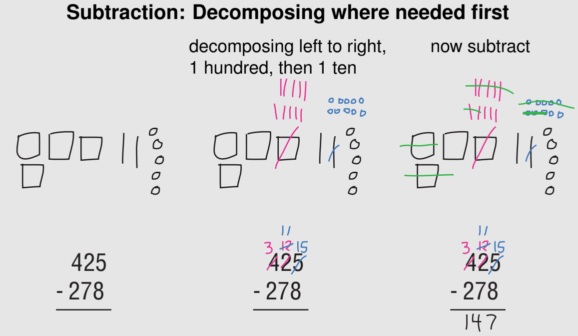
Addition proceeds from left to right, but could also have gone from right to left. There are two advantages of working left to right: Many students prefer it because they read from left to right, and working first with the largest units yields a closer approximation earlier.

Addition: Recording newly composed units on the same line

456 +167	456 +167 1 3	456 +167 11 23	456 +167 11 623
	Add the ones, $6+7$, and record these 13 ones with 3 in the ones place and 1 on the line under the tens column.	Add the tens, $5+6+1$, and record these 12 tens with 2 in the tens place and 1 on the line under the hundreds column.	Add the hundreds, $4+1+1$ and record these 6 hundreds in the hundreds column.

Digits representing newly composed units are placed below the addends. This placement has several advantages. Each two-digit partial sum (e.g., "13") is written with the digits close to each other, suggesting their origin. In "adding from the top down," usually sums of larger digits are computed first, and the easy-to-add "1" is added to that sum, freeing students from holding an altered digit in memory. The original numbers are not changed by adding numbers to the first addend; three multi-digit numbers (the addends and the total) can be seen clearly. It is easier to write teen numbers in their usual order (e.g., as 1 then 3) rather than "write the 3 and carry the 1" (write 3, then 1).

Subtraction: Decomposing where needed first



All necessary decomposing is done first, then the subtractions are carried out. This highlights the two major steps involved and can help to inhibit the common error of subtracting a smaller digit on the top from a larger digit. Decomposing and subtracting can start from the left (as shown) or the right.

ple, they might say, "100 and 400 is 500. And 70 and 30 is another hundred, so 600. Then 8, 9, 10, 11 …and the other 10 is 21. So, 621." Keeping track of what is being added is easier using a written form of such reasoning and makes it easier to discuss. There are two kinds of decompositions in this strategy. Both addends are decomposed into hundreds, tens, and ones, and the first addend is decomposed successively into the part already added and the part still to add.

Students should continue to develop proficiency with mental computation. They mentally add 10 or 100 to a given number between 100 and 900, and mentally subtract 10 or 100 from a given number between 100 and 900.[2.NBT.8]

2.NBT.8 Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900.

Grade 3

At Grade 3, the major focus is multiplication,• so students' work with addition and subtraction is limited to maintenance of fluency within 1000 for some students and building fluency to within 1000 for others.

• See the progression on Operations and Algebraic Thinking.

Use place value understanding and properties of operations to perform multi-digit arithmetic Students continue adding and subtracting within 1000.[3.NBT.2] They achieve fluency with strategies and algorithms that are based on place value, properties of operations, and/or the relationship between addition and subtraction. Such fluency can serve as preparation for learning standard algorithms in Grade 4, if the computational methods used can be connected with those algorithms.

3.NBT.2 Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.

Students use their place value understanding to round numbers to the nearest 10 or 100.[3.NBT.1] They need to understand that when moving to the right across the places in a number (e.g., 456), the digits represent smaller units. When rounding to the nearest 10 or 100, the goal is to approximate the number by the closest number with no ones or no tens and ones (e.g., so 456 to the nearest ten is 460; and to the nearest hundred is 500). Rounding to the unit represented by the leftmost place is typically the sort of estimate that is easiest for students. Rounding to the unit represented by a place in the middle of a number may be more difficult for students (the surrounding digits are sometimes distracting). Rounding two numbers before computing can take as long as just computing their sum or difference.

3.NBT.1 Use place value understanding to round whole numbers to the nearest 10 or 100.

The special role of 10 in the base-ten system is important in understanding multiplication of one-digit numbers with multiples of 10.[3.NBT.3] For example, the product 3×50 can be represented as 3 groups of 5 tens, which is 15 tens, which is 150. This reasoning relies on the associative property of multiplication: $3 \times 50 = 3 \times (5 \times 10) = (3 \times 5) \times 10 = 15 \times 10 = 150$. It is an example of how to explain an instance of a calculation pattern for these products: calculate the product of the non-zero digits, then shift the product one place to the left to make the result ten times as large.•

3.NBT.3 Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9×80, 5×60) using strategies based on place value and properties of operations.

• **Grade 3 explanations for "15 tens is 150"**

- Skip-counting by 50. 5 tens is 50, 100, 150.
- Counting on by 5 tens. 5 tens is 50, 5 more tens is 100, 5 more tens is 150.
- Decomposing 15 tens. 15 tens is 10 tens and 5 tens. 10 tens is 100. 5 tens is 50. So 15 tens is 100 and 50, or 150.
- Decomposing 15.

$$\begin{aligned} 15 \times 10 &= (10+5) \times 10 \\ &= (10 \times 10) + (5 \times 10) \\ &= 100 + 50 \\ &= 150 \end{aligned}$$

All of these explanations are correct. However, skip-counting and counting on become more difficult to use accurately as numbers become larger, e.g., in computing 5×90 or explaining why 45 tens is 450, and needs modification for products such as 4×90. The first does not indicate any place value understanding.

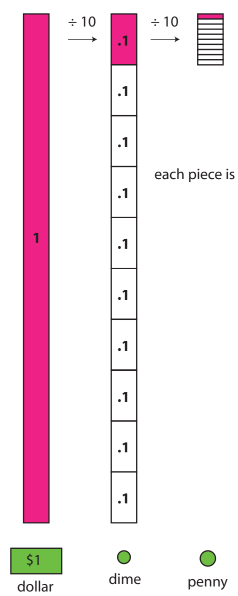
Grade 4

At Grade 4, students extend their work in the base-ten system. They use standard algorithms to fluently add and subtract. They use methods based on place value and properties of operations supported by suitable representations to multiply and divide with multi-digit numbers.

Generalize place value understanding for multi-digit whole numbers In the base-ten system, the value of each place is 10 times the value of the place to the immediate right.[4.NBT.1] Because of this, multiplying by 10 yields a product in which each digit of the multiplicand is shifted one place to the left.

To read numerals between 1,000 and 1,000,000, students need to understand the role of commas. Each sequence of three digits made by commas is read as hundreds, tens, and ones, followed by the name of the appropriate base-thousand unit (thousand, million, billion, trillion, etc.). Thus, 457,000 is read "four hundred fifty seven thousand."[4.NBT.2] The same methods students used for comparing and rounding numbers in previous grades apply to these numbers, because of the uniformity of the base-ten system.

Decimal notation and fractions Students in Grade 4 work with fractions having denominators 10 and 100.[4.NF.5] Because it involves partitioning into 10 equal parts and treating the parts as numbers called one tenth and one hundredth, work with these fractions can be used as preparation to extend the base-ten system to non-whole numbers.

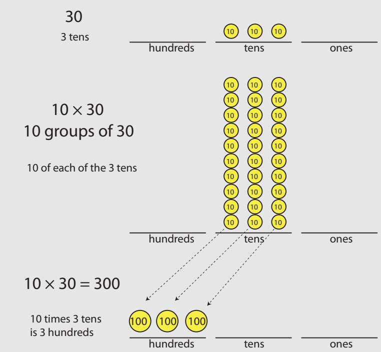


Using the unit fractions $\frac{1}{10}$ and $\frac{1}{100}$, non-whole numbers like $23\frac{7}{10}$ can be written in an expanded form that extends the form used with whole numbers: $2 \times 10 + 3 \times 1 + 7 \times \frac{1}{10}$.[4.NF.4b] As with whole-number expansions in the base-ten system, each unit in this decomposition is ten times the unit to its right. This can be connected with the use of base-ten notation to represent $2\times10+3\times1+7\times\frac{1}{10}$ as 23.7. Using decimals allows students to apply familiar place value reasoning to fractional quantities.[4.NF.6] The Number and Operations—Fractions Progression discusses decimals to hundredths and comparison of decimals[4.NF.7] in more detail.

The decimal point is used to signify the location of the ones place, but its location may suggest there should be a "oneths" place to its right in order to create symmetry with respect to the decimal point.

4.NBT.1 Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right.

10×30 represented as 3 tens each taken 10 times

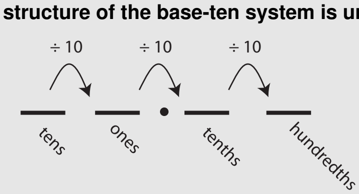


Each of the 3 tens becomes a hundred and moves to the left. In the product, the 3 in the tens place of 30 is shifted one place to the left to represent 3 hundreds. In 300 divided by 10 the 3 is shifted one place to the right in the quotient to represent 3 tens.

4.NBT.2 Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using $>$, =, and $<$ symbols to record the results of comparisons.

4.NF.5 Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.[2]

The structure of the base-ten system is uniform



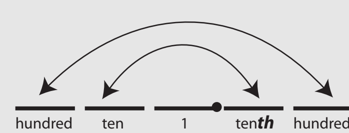
4.NF.4b Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

- b Understand a multiple of a/b as a multiple of $1/b$, and use this understanding to multiply a fraction by a whole number.

4.NF.6 Use decimal notation for fractions with denominators 10 or 100.

4.NF.7 Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols $>$, =, or $<$, and justify the conclusions, e.g., by using a visual model.

Symmetry with respect to the ones place



However, because one is the basic unit from which the other base-ten units are derived, the symmetry occurs instead with respect to the ones place.

Ways of reading decimals aloud vary. Mathematicians and scientists often read 0.15 aloud as "zero point one five" or "point one five." (Decimals smaller than one may be written with or without a zero before the decimal point.) Decimals with many non-zero digits are more easily read aloud in this manner. (For example, the number π, which has infinitely many non-zero digits, begins $3.1415\ldots$.)

Other ways to read 0.15 aloud are "1 tenth and 5 hundredths" and "15 hundredths," just as 1,500 is sometimes read "15 hundred" or "1 thousand, 5 hundred." Similarly, 150 is read "one hundred and fifty" or "a hundred fifty" and understood as 15 tens, as 10 tens and 5 tens, and as $100 + 50$.

Just as 15 is understood as 15 ones and as 1 ten and 5 ones in computations with whole numbers, 0.15 is viewed as 15 hundredths and as 1 tenth and 5 hundredths in computations with decimals.

It takes time to develop understanding and fluency with the different forms. Layered cards for decimals can help students become fluent with decimal equivalencies such as three tenths is thirty hundredths.

Use place value understanding and properties of operations to perform multi-digit arithmetic At Grade 4, students become fluent with the standard addition and subtraction algorithms.[4.NBT.4] As discussed at the beginning of this progression, these algorithms rely on adding or subtracting like base-ten units (ones with ones, tens with tens, hundreds with hundreds, and so on) and composing or decomposing base-ten units as needed (such as composing 10 ones to make 1 ten or decomposing 1 hundred to make 10 tens). In mathematics, an algorithm is defined by its steps and not by the way those steps are recorded in writing. With this in mind, minor variations in methods of recording standard algorithms are acceptable.

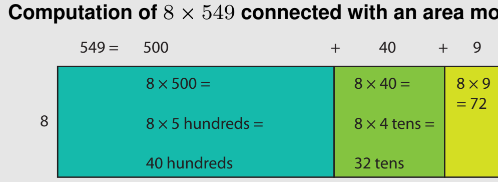
In fourth grade, students compute products of one-digit numbers and multi-digit numbers (up to four digits) and products of two two-digit numbers.[4.NBT.5] They divide multi-digit numbers (up to four digits) by one-digit numbers. As with addition and subtraction, students should use methods they understand and can explain. Visual representations such as area and array diagrams that students draw and connect to equations and other written numerical work are useful for this purpose. By reasoning repeatedly about the connection between math drawings and written numerical work, students can come to see multiplication and division algorithms as abbreviations or summaries of their reasoning about quantities.

Students can invent and use fast special strategies while also working towards understanding general methods and the standard algorithm.

4.NBT.4 Fluently add and subtract multi-digit whole numbers using the standard algorithm.

4.NBT.5 Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Computation of 8×549 connected with an area model



Each part of the region above corresponds to one of the terms in the computation below.

$$\begin{aligned} 8 \times 549 &= 8 \times (500 + 40 + 9) \\ &= 8 \times 500 + 8 \times 40 + 8 \times 9. \end{aligned}$$

This can also viewed as finding how many objects are in 8 groups of 549 objects, by finding the cardinalities of 8 groups of 500, 8 groups of 40, and 8 groups of 9, then adding them.

One component of understanding general methods for multiplication is understanding how to compute products of one-digit numbers and multiples of 10, 100, and 1000. This extends work in Grade 3 on products of one-digit numbers and multiples of 10. We can calculate 6×700 by calculating 6×7 and then shifting the result to the left two places (by placing two zeros at the end to show that these are hundreds) because 6 groups of 7 hundred is 6×7 hundreds, which is 42 hundreds, or 4,200. Students can use this place value reasoning, which can also be supported with diagrams of arrays or areas, as they develop and practice using the patterns in relationships among products such as 6×7, 6×70, 6×700, and 6×7000. Products of 5 and even numbers, such as 5×4, 5×40, 5×400, 5×4000 and 4×5, 4×50, 4×500, 4×5000 might be discussed and practiced separately afterwards because they may seem at first to violate the patterns by having an "extra" 0 that comes from the one-digit product.

Another part of understanding general base-ten methods for multi-digit multiplication is understanding the role played by the distributive property. This allows numbers to be decomposed into base-ten units, products of the units to be computed, then combined. By decomposing the factors into like base-ten units and applying the distributive property, multiplication computations are reduced to single-digit multiplications and products of numbers with multiples of 10, of 100, and of 1000. Students can connect diagrams of areas or arrays to numerical work to develop understanding of general base-ten multiplication methods.

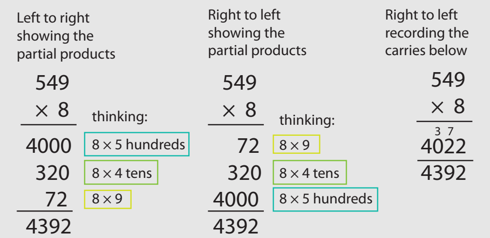
Computing products of two two-digit numbers requires using the distributive property several times when the factors are decomposed into base-ten units. For example,

$$\begin{aligned} 36 \times 94 &= (30+6) \times (90+4) \\ &= (30+6) \times 90 + (30+6) \times 4 \\ &= 30 \times 90 + 6 \times 90 + 30 \times 4 + 6 \times 4. \end{aligned}$$

General methods for computing quotients of multi-digit numbers and one-digit numbers rely on the same understandings as for multiplication, but cast in terms of division.[4.NBT.6] One component is quotients of multiples of 10, 100, or 1000 and one-digit numbers. For example, $42 \div 6$ is related to $420 \div 6$ and $4200 \div 6$. Students can draw on their work with multiplication and they can also reason that $4200 \div 6$ means partitioning 42 hundreds into 6 equal groups, so there are 7 hundreds in each group.

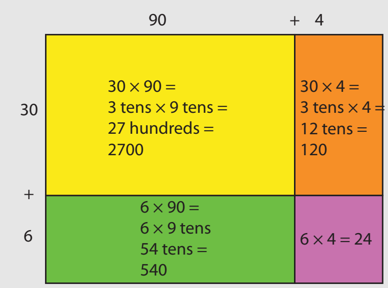
Another component of understanding general methods for multi-digit division computation is the idea of decomposing the dividend into like base-ten units and finding the quotient unit by unit, starting with the largest unit and continuing on to smaller units. As with multiplication, this relies on the distributive property. This can be viewed as finding the side length of a rectangle (the divisor is the length of the other side) or as allocating objects (the divisor is the number of groups). See the figures on the next page for examples.

Computation of 8×549: Ways to record general methods



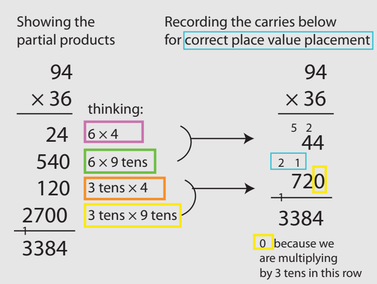
The first method proceeds from left to right, and the others from right to left. In the third method, the digits representing new units are written below the line rather than above 549, thus keeping the digits of the products close to each other, e.g., the 7 from $8 \times 9 = 72$ is written diagonally to the left of the 2 rather than above the 4 in 549.

Computation of 36×94 connected with an area model



The products of like base-ten units are shown as parts of a rectangular region.

Computation of 36×94: Ways to record general methods



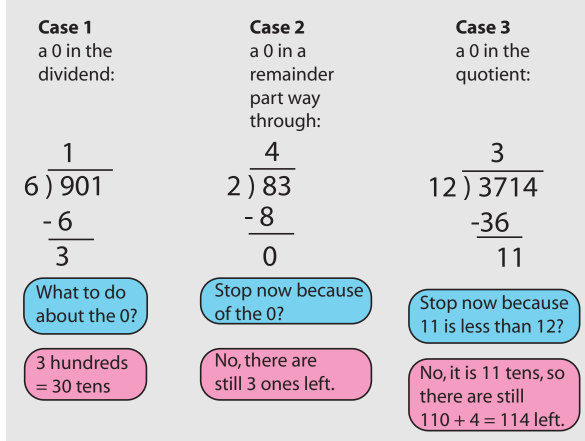
These proceed from right to left, but could go left to right. On the right, digits that represent newly composed tens and hundreds are written below the line instead of above 94. The digits 2 and 1 are surrounded by a blue box. The 1 from $30 \times 4 = 120$ is placed correctly in the hundreds place and the digit 2 from $30 \times 90 = 2700$ is placed correctly in the thousands place. If these digits had been placed above 94, they would be in incorrect places. Note that the 0 (surrounded by a yellow box) in the ones place of the second line of the method on the right is there because the whole line of digits is produced by multiplying by 30 (not 3).

4.NBT.6 Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

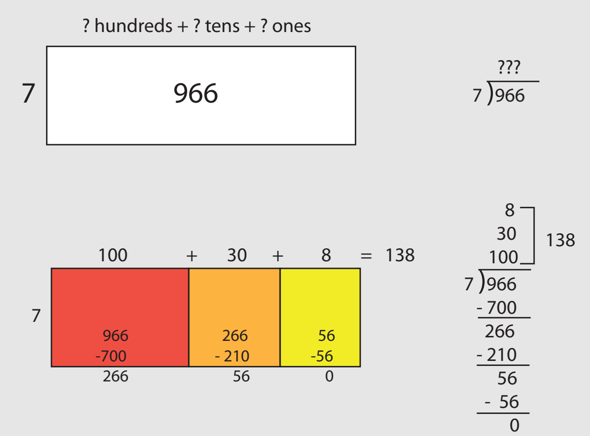
Multi-digit division requires working with remainders. In preparation for working with remainders, students can compute sums of a product and a number, such as $4 \times 8 + 3$. In multi-digit division, students will need to find the greatest multiple less than a given number. For example, when dividing by 6, the greatest multiple of 6 less than 50 is $6 \times 8 = 48$. Students can think of these "greatest multiples" in terms of putting objects into groups. For example, when 50 objects are shared among 6 groups, the largest whole number of objects that can be put in each group is 8, and 2 objects are left over. (Or when 50 objects are allocated into groups of 6, the largest whole number of groups that can be made is 8, and 2 objects are left over.) The equation $6 \times 8 + 2 = 50$ (or $8 \times 6 + 2 = 50$) corresponds with this situation.

Cases involving 0 in division may require special attention.

Cases involving 0 in division

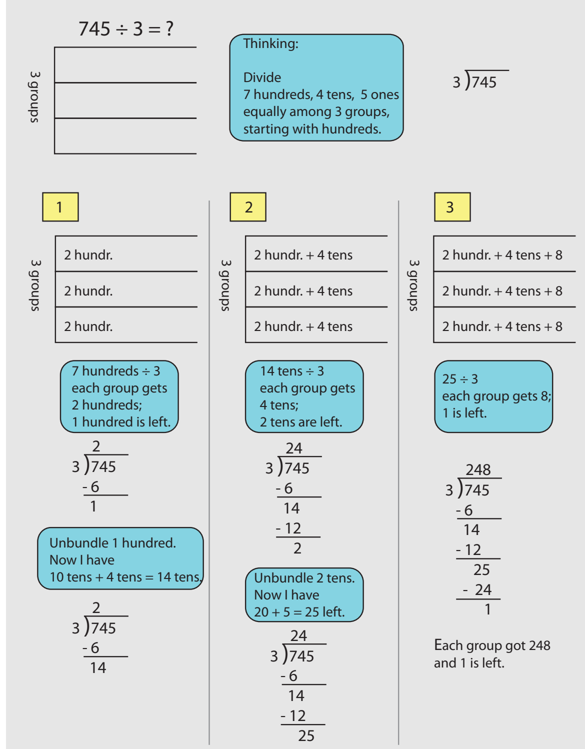


Division as finding side length



$966 \div 7$ is viewed as finding the unknown side length of a rectangular region with area 966 square units and a side of length 7 units. The amount of hundreds is found, then tens, then ones. This yields a decomposition into three regions of dimensions 7 by 100, 7 by 30, and 7 by 8. It can be connected with the decomposition of 966 as $7 \times 100 + 7 \times 30 + 7 \times 8$. By the distributive property, this is $7 \times (100 + 30 + 8)$, so the unknown side length is 138. In the recording on the right, amounts of hundreds, tens, and ones are represented by numbers rather than by digits, e.g., 700 instead of 7.

Division as finding group size



$745 \div 3$ can be viewed as allocating 745 objects bundled in 7 hundreds, 4 tens, and 3 ones equally among 3 groups. In Step 1, the 2 indicates that each group got 2 hundreds, the 6 is the number of hundreds allocated, and the 1 is the number of hundreds not allocated. After Step 1, the remaining hundred is decomposed as 10 tens and combined with the 4 tens (in 745) to make 14 tens.

Grade 5

In Grade 5, students extend their understanding of the base-ten system to decimals to the thousandths place, building on their Grade 4 work with tenths and hundredths. They become fluent with the standard multiplication algorithm with multi-digit whole numbers. They reason about dividing whole numbers with two-digit divisors, and reason about adding, subtracting, multiplying, and dividing decimals to hundredths.

Understand the place value system Students extend their understanding of the base-ten system to the relationship between adjacent places, how numbers compare, and how numbers round for decimals to thousandths.

New at Grade 5 is the use of whole number exponents to denote powers of 10.[5.NBT.2] Students understand why multiplying by a power of 10 shifts the digits of a whole number or decimal that many places to the left. For example, multiplying by 10^4 is multiplying by 10 four times. Multiplying by 10 once shifts every digit of the multiplicand one place to the left in the product (the product is ten times as large) because in the base-ten system the value of each place is 10 times the value of the place to its right. So multiplying by 10 four times shifts every digit 4 places to the left. Patterns in the number of 0s in products of a whole numbers and a power of 10 and the location of the decimal point in products of decimals with powers of 10 can be explained in terms of place value. Because students have developed their understandings of and computations with decimals in terms of multiples (consistent with 4.OA.4) rather than powers, connecting the terminology of multiples with that of powers affords connections between understanding of multiplication and exponentiation.

[5.NBT.2] Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.

Perform operations with multi-digit whole numbers and with decimals to hundredths At Grade 5, students fluently compute products of whole numbers using the standard algorithm.[5.NBT.5] Underlying this algorithm are the properties of operations and the base-ten system (see the Grade 4 section).

[5.NBT.5] Fluently multiply multi-digit whole numbers using the standard algorithm.

Division strategies in Grade 5 involve breaking the dividend apart into like base-ten units and applying the distributive property to find the quotient place by place, starting from the highest place. (Division can also be viewed as finding an unknown factor: the dividend is the product, the divisor is the known factor, and the quotient is the unknown factor.) Students continue their fourth grade work on division, extending it to computation of whole number quotients with dividends of up to four digits and two-digit divisors. Estimation becomes relevant when extending to two-digit divisors. Even if students round appropriately, the resulting estimate may need to be adjusted.

Recording division after an underestimate

1655 ÷ 27	1	
	10	61
Rounding 27	(30) 50	
to 30 produces	27) 1655	
the underestimate	-1350	
50 at the first step	305	
but this method	-270	
allows the division	35	
process to be	-27	
continued	8	

Draft, 4/21/2012, comment at `commoncoretools.wordpress.com`.

Because of the uniformity of the structure of the base-ten system, students use the same place value understanding for adding and subtracting decimals that they used for adding and subtracting whole numbers.[5.NBT.7] Like base-ten units must be added and subtracted, so students need to attend to aligning the corresponding places correctly (this also aligns the decimal points). It can help to put 0s in places so that all numbers show the same number of places to the right of the decimal point. Although whole numbers are not usually written with a decimal point, but that a decimal point with 0s on its right can be inserted (e.g., 16 can also be written as 16.0 or 16.00). The process of composing and decomposing a base-ten unit is the same for decimals as for whole numbers and the same methods of recording numerical work can be used with decimals as with whole numbers. For example, students can write digits representing new units below on the addition or subtraction line, and they can decompose units wherever needed before subtracting.

General methods used for computing products of whole numbers extend to products of decimals. Because the expectations for decimals are limited to thousandths and expectations for factors are limited to hundredths at this grade level, students will multiply tenths with tenths and tenths with hundredths, but they need not multiply hundredths with hundredths. Before students consider decimal multiplication more generally, they can study the effect of multiplying by 0.1 and by 0.01 to explain why the product is ten or a hundred times as small as the multiplicand (moves one or two places to the right). They can then extend their reasoning to multipliers that are single-digit multiples of 0.1 and 0.01 (e.g., 0.2 and 0.02, etc.).

There are several lines of reasoning that students can use to explain the placement of the decimal point in other products of decimals. Students can think about the product of the smallest base-ten units of each factor. For example, a tenth times a tenth is a hundredth, so 3.2×7.1 will have an entry in the hundredth place. Note, however, that students might place the decimal point incorrectly for 3.2×8.5 unless they take into account the 0 in the ones place of 32×85. (Or they can think of 0.2×0.5 as 10 hundredths.) They can also think of the decimals as fractions or as whole numbers divided by 10 or 100.[5.NF.3] When they place the decimal point in the product, they have to divide by a 10 from each factor or 100 from one factor. For example, to see that $0.6 \times 0.8 = 0.48$, students can use fractions: $\frac{6}{10} \times \frac{8}{10} = \frac{48}{100}$.[5.NF.4] Students can also reason that when they carry out the multiplication without the decimal point, they have multiplied each decimal factor by 10 or 100, so they will need to divide by those numbers in the end to get the correct answer. Also, students can use reasoning about the sizes of numbers to determine the placement of the decimal point. For example, 3.2×8.5 should be close to 3×9, so 27.2 is a more reasonable product for 3.2×8.5 than 2.72 or 272. This estimation-based method is not reliable in all cases, however, especially in cases students will encounter in later grades. For example, it is not easy to decide where to place

5.NBT.7 Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

5.NF.3 Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

5.NF.4 Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

the decimal point in 0.023×0.0045 based on estimation. Students can summarize the results of their reasoning such as those above as specific numerical patterns and then as one general overall pattern such as "the number of decimal places in the product is the sum of the number of decimal places in each factor."

General methods used for computing quotients of whole numbers extend to decimals with the additional issue of placing the decimal point in the quotient. As with decimal multiplication, students can first examine the cases of dividing by 0.1 and 0.01 to see that the quotient becomes 10 times or 100 times as large as the dividend (see also the Number and Operations—Fractions Progression). For example, students can view $7 \div 0.1 = \square$ as asking how many tenths are in 7.[5.NF.7b] Because it takes 10 tenths make 1, it takes 7 times as many tenths to make 7, so $7 \div 0.1 = 7 \times 10 = 70$. Or students could note that 7 is 70 tenths, so asking how many tenths are in 7 is the same as asking how many tenths are in 70 tenths, which is 70. In other words, $7 \div 0.1$ is the same as $70 \div 1$. So dividing by 0.1 moves the number 7 one place to the left, the quotient is ten times as big as the dividend. As with decimal multiplication, students can then proceed to more general cases. For example, to calculate $7 \div 0.2$, students can reason that 0.2 is 2 tenths and 7 is 70 tenths, so asking how many 2 tenths are in 7 is the same as asking how many 2 tenths are in 70 tenths. In other words, $7 \div 0.2$ is the same as $70 \div 2$; multiplying both the 7 and the 0.2 by 10 results in the same quotient. Or students could calculate $7 \div 0.2$ by viewing 0.2 as 2×0.1, so they can first divide 7 by 2, which is 3.5, and then divide that result by 0.1, which makes 3.5 ten times as large, namely 35. Dividing by a decimal less than 1 results in a quotient larger than the dividend[5.NF.5] and moves the digits of the dividend one place to the left. Students can summarize the results of their reasoning as specific numerical patterns then as one general overall pattern such as "when the decimal point in the divisor is moved to make a whole number, the decimal point in the dividend should be moved the same number of places."

5.NF.7b Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.[3]

b Interpret division of a whole number by a unit fraction, and compute such quotients.

5.NF.5 Interpret multiplication as scaling (resizing), by:

a Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

b Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

Extending beyond Grade 5

At Grade 6, students extend their fluency with the standard algorithms, using these for all four operations with decimals and to compute quotients of multi-digit numbers. At Grade 6 and beyond, students may occasionally compute with numbers larger than those specified in earlier grades as required for solving problems, but the Standards do not specify mastery with such numbers.

In Grade 6, students extend the base-ten system to negative numbers. In Grade 7, they begin to do arithmetic with such numbers.

By reasoning about the standard division algorithm, students learn in Grade 7 that every fraction can be represented with a decimal that either terminates or repeats. In Grade 8, students learn informally that every number has a decimal expansion, and that those with a terminating or repeating decimal representation are rational numbers (i.e., can be represented as a quotient of integers). There are numbers that are not rational (irrational numbers), such as the square root of 2. (It is not obvious that the square root of 2 is not rational, but this can be proved.) In fact, surprisingly, it turns out that most numbers are not rational. Irrational numbers can always be approximated by rational numbers.

In Grade 8, students build on their work with rounding and exponents when they begin working with scientific notation. This allows them to express approximations of very large and very small numbers compactly by using exponents and generally only approximately by showing only the most significant digits. For example, the Earth's circumference is approximately 40,000,000 m. In scientific notation, this is 4×10^7 m.

The Common Core Standards are designed so that ideas used in base-ten computation, as well as in other domains, can support later learning. For example, use of the distributive property occurs together with the idea of combining like units in the NBT and NF standards. Students use these ideas again when they calculate with polynomials in high school.

The distributive property and like units: Multiplication of whole numbers and polynomials

52×73	$(5x + 2)(7x + 3)$	
$= (5 \times 10 + 2)(7 \times 10 + 3)$	$= (5x + 2)(7x + 3)$	decomposing as like units (powers of 10 or powers of x)
$= 5 \times 10(7 \times 10 + 3) + 2 \times (7 \times 10 + 3)$	$= 5x(7x + 3) + 2(7x + 3)$	using the distributive property
$= 35 \times 10^2 + 15 \times 10 + 14 \times 10 + 2 \times 3$	$= 35x^2 + 15x + 14x + 2 \times 3$	using the distributive property again
$= 35 \times 10^2 + 29 \times 10 + 6$	$= 35x^2 + 29x + 6$	combining like units (powers of 10 or powers of x)

Progressions for the Common Core State Standards in Mathematics (draft)

22 April 2011

6–8, Expressions and Equations

Overview

An *expression* expresses something. Facial expressions express emotions. Mathematical expressions express calculations with numbers. Some of the numbers might be given explicitly, like 2 or $\frac{3}{4}$. Other numbers in the expression might be represented by letters, such as x, y, P, or n. The calculation an expression represents might use only a single operation, as in $4+3$ or $3x$, or it might use a series of nested or parallel operations, as in $3(a+9)-9/b$. An expression can consist of just a single number, even 0.

Letters standing for numbers in an expression are called *variables*. In good practice, including in student writing, the meaning of a variable is specified by the surrounding text; an expression by itself gives no intrinsic meaning to the variables in it. Depending on the context, a variable might stand for a specific number, for example the solution to a word problem; it might be used in a universal statement true for all numbers, for example when we say that that $a+b=b+a$ for all numbers a and b; or it might stand for a range of numbers, for example when we say that $\sqrt{x^2}=x$ for $x>0$. In choosing variables to represent quantities, students specify a unit; rather than saying "let G be gasoline," they say "let G be the number of gallons of gasoline".[MP6]

[MP6] Be precise in defining variables.

An expression is a phrase in a sentence about a mathematical or real-world situation. As with a facial expression, however, you can read a lot from an algebraic expression (an expression with variables in it) without knowing the story behind it, and it is a goal of this progression for students to see expressions as objects in their own right, and to read the general appearance and fine details of algebraic expressions.

An *equation* is a statement that two expressions are equal, such as $10+0.02n=20$, or $3+x=4+x$, or $2(a+1)=2a+2$. It is an important aspect of equations that the two expressions on either side of the equal sign might not actually always be equal; that is, the equation might be a true statement for some values of the variable(s) and a false statement for others. For example,

$10 + 0.02n = 20$ is true only if $n = 500$; and $3 + x = 4 + x$ is not true for any number x; and $2(a + 1) = 2a + 2$ is true for all numbers a. A *solution* to an equation is a number that makes the equation true when substituted for the variable (or, if there is more than one variable, it is a number for each variable). An equation may have no solutions (e.g., $3 + x = 4 + x$ has no solutions because, no matter what number x is, it is not true that adding 3 to x yields the same answer as adding 4 to x). An equation may also have every number for a solution (e.g., $2(a + 1) = 2a + 2$). An equation that is true no matter what number the variable represents is called an *identity*, and the expressions on each side of the equation are said to be *equivalent expressions*. For example $2(a + 1)$ and $2a + 2$ are equivalent expressions. In Grades 6–8, students start to use properties of operations to manipulate algebraic expressions and produce different but equivalent expressions for different purposes. This work builds on their extensive experience in K–5 working with the properties of operations in the context of operations with whole numbers, decimals and fractions.

Grade 6

Apply and extend previous understandings of arithmetic to algebraic expressions Students have been writing numerical expressions since Kindergarten, such as

$$2+3 \qquad 7+6+3 \qquad 4\times(2\times 3)$$

$$8\times 5+8\times 2 \qquad \frac{1}{3}(8+7+3) \qquad \frac{3}{\frac{1}{2}}.$$

In Grade 5 they used whole number exponents to express powers of 10, and in Grade 6 they start to incorporate whole number exponents into numerical expressions, for example when they describe a square with side length 50 feet as having an area of 50^2 square feet.[6.EE.1]

Students have also been using letters to represent an unknown quantity in word problems since Grade 3. In Grade 6 they begin to work systematically with algebraic expressions. They express the calculation "Subtract y from 5" as $5-y$, and write expressions for repeated numerical calculations.[MP8] For example, students might be asked to write a numerical expression for the change from a $10 bill after buying a book at various prices:

Price of book ($)	5.00	6.49	7.15
Change from $10	$10-5$	$10-6.49$	$10-7.15$

Abstracting the pattern they write $10-p$ for a book costing p dollars, thus summarizing a calculation that can be carried out repeatedly with different numbers.[6.EE.2a] Such work also helps students interpret expressions. For example, if there are 3 loose apples and 2 bags of A apples each, students relate quantities in the situation to the terms in the expression $3+2A$.

As they start to solve word problems algebraically, students also use more complex expressions. For example, in solving the word problem

> Daniel went to visit his grandmother, who gave him $5.50. Then he bought a book costing $9.20. If he has $2.30 left, how much money did he have before visiting his grandmother?

students might obtain the expression $x+5.50-9.20$ by following the story forward, and then solve the equation $x+5.50-9.20=2.30$.• Students may need explicit guidance in order to develop the strategy of working forwards, rather than working backwards from the 2.30 and calculating $2.30+9.20-5.50$.[6.EE.7] As word problems get more complex, students find greater benefit in representing the problem algebraically by choosing variables to represent quantities, rather than attempting a direct numerical solution, since the former approach provides general methods and relieves demands on working memory.

6.EE.1 Write and evaluate numerical expressions involving whole-number exponents.

MP8 Look for regularity in a repeated calculation and express it with a general formula.

6.EE.2a Write, read, and evaluate expressions in which letters stand for numbers.

a Write expressions that record operations with numbers and with letters standing for numbers.

• Notice that in this problem, like many problems, a quantity, "money left," is expressed in two distinct ways:

1. starting amount + amount from grandma − amount spent
2. $2.30

Because these two expressions refer to the same quantity in the problem situation, they are equal to each other. The equation formed by representing their equality can then be solved to find the unknown value (that is, the value of the variable that makes the equation fit the situation).

6.EE.7 Solve real-world and mathematical problems by writing and solving equations of the form $x+p=q$ and $px=q$ for cases in which p, q and x are all nonnegative rational numbers.

Students in Grade 5 began to move from viewing expressions as actions describing a calculation to viewing them as objects in their own right;[5.OA.2] in Grade 6 this work continues and becomes more sophisticated. They describe the structure of an expression, seeing $2(8+7)$ for example as a product of two factors the second of which, $(8+7)$, can be viewed as both a single entity and a sum of two terms. They interpret the structure of an expression in terms of a context: if a runner is $7t$ miles from her starting point after t hours, what is the meaning of the 7?[MP7] If a, b, and c are the heights of three students in inches, they recognize that the coefficient $\frac{1}{3}$ in $\frac{1}{3}(a+b+c)$ has the effect of reducing the size of the sum, and they also interpret multiplying by $\frac{1}{3}$ as dividing by 3.[6.EE.2b] Both interpretations are useful in connection with understanding the expression as the mean of a, b, and c.[6.SP.3]

In the work on number and operations in Grades K–5, students have been using properties of operations to write expressions in different ways. For example, students in grades K–5 write $2+3=3+2$ and $8\times5+8\times2=8\times(5+2)$ and recognize these as instances of general properties which they can describe. They use the "any order, any grouping" property• to see the expression $7+6+3$ as $(7+3)+6$, allowing them to quickly evaluate it. The properties are powerful tools that students use to accomplish what they want when working with expressions and equations. They can be used at any time, in any order, whenever they serve a purpose.

Work with numerical expressions prepares students for work with algebraic expressions. During the transition, it can be helpful for them to solve numerical problems in which it is more efficient to hold numerical expressions unevaluated at intermediate steps. For example, the problem

> Fred and George Weasley make 150 "Deflagration Deluxe" boxes of Weasleys' Wildfire Whiz-bangs at a cost of 17 Galleons each, and sell them for 20 Galleons each. What is their profit?

is more easily solved by leaving unevaluated the total cost, 150×17 Galleons, and the total revenue 150×20 Galleons, until the subtraction step, where the distributive law can be used to calculate the answer as $150\times20-150\times17=150\times3=450$ Galleons. A later algebraic version of the problem might ask for the sale price that will yield a given profit, with the sale price represented by a letter such as p. The habit of leaving numerical expressions unevaluated prepares students for constructing the appropriate algebraic equation to solve such a problem.

As students move from numerical to algebraic work the multiplication and division symbols $\times$ and $\div$ are replaced by the conventions of algebraic notation. Students learn to use either a dot for multiplication, e.g., $1\cdot2\cdot3$ instead of $1\times2\times3$, or simple juxtaposition, e.g., $3x$ instead of $3\times x$ (during the transition, students may indicate all multiplications with a dot, writing $3\cdot x$ for $3x$). A firm grasp

5.OA.2 Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them.

MP7 Looking for structure in expressions by parsing them into a sequence of operations; making use of the structure to interpret the expression's meaning in terms of the quantities represented by the variables.

6.EE.2b Write, read, and evaluate expressions in which letters stand for numbers.

b Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity.

6.SP.3 Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.

• The "any order, any grouping" property is a combination of the commutative and associative properties. It says that sequence of additions and subtractions may be calculated in any order, and that terms may be grouped together any way.

Some common student difficulties

- Failure to see juxtaposition as indicating multiplication, e.g., evaluating $3x$ as 35 when $x=5$, or rewriting $8-2a$ as $6a$
- Failure to see hidden 1s, rewriting $4C-C$ as 4 instead of seeing $4C-C$ as $4\cdot C-1\cdot C$ which is $3\cdot C$.

on variables as numbers helps students extend their work with the properties of operations from arithmetic to algebra.[MP2] For example, students who are accustomed to mentally calculating 5×37 as $5 \times (30 + 7) = 150 + 35$ can now see that $5(3a + 7) = 15a + 35$ for all numbers a. They apply the distributive property to the expression $3(2 + x)$ to produce the equivalent expression $6 + 3x$ and to the expression $24x + 18y$ to produce the equivalent expression $6(4x + 3y)$.[6.EE.3]

Students evaluate expressions that arise from formulas used in real-world problems, such as the formulas $V = s^3$ and $A = 6s^2$ for the volume and surface area of a cube. In addition to using the properties of operations, students use conventions about the order in which arithmetic operations are performed in the absence of parentheses.[6.EE.2c] It is important to distinguish between such conventions, which are notational conveniences that allow for algebraic expressions to be written with fewer parentheses, and properties of operations, which are fundamental properties of the number system and undergird all work with expressions. In particular, the mnemonic PEMDAS• can mislead students into thinking, for example, that addition must always take precedence over subtraction because the A comes before the S, rather than the correct convention that addition and subtraction proceed from left to right (as do multiplication and division). This can lead students to make mistakes such as simplifying $n - 2 + 5$ as $n - 7$ (instead of the correct $n + 3$) because they add the 2 and the 5 before subtracting from n.[6.EE.4]

The order of operations tells us how to interpret expressions, but does not necessarily dictate how to calculate them. For example, the P in PEMDAS indicates that the expression $8 \times (5 + 1)$ is to be interpreted as 8 times a number which is the sum of 5 and 1. However, it does not dictate the expression must be calculated this way. A student might well see it, through an implicit use of the distributive law, as $8 \times 5 + 8 \times 1 = 40 + 8 = 48$.

The distributive law is of fundamental importance. Collecting like terms, e.g., $5b + 3b = (5 + 3)b = 8b$, should be seen as an application of the distributive law, not as a separate method.

Reason about and solve one-variable equations and inequalities

In Grades K–5 students have been writing numerical equations and simple equations involving one operation with a variable. In Grade 6 they start the systematic study of equations and inequalities and methods of solving them. Solving is a process of reasoning to find the numbers which make an equation true, which can include checking if a given number is a solution.[6.EE.5] Although the process of reasoning will eventually lead to standard methods for solving equations, students should study examples where looking for structure pays off, such as in $4x + 3x = 3x + 20$, where they can see that $4x$ must be 20 to make the two sides equal.

This understanding can be reinforced by comparing arithmetic

MP2 Connect abstract symbols to their numerical referents.

6.EE.3 Apply the properties of operations to generate equivalent expressions.

6.EE.2c Write, read, and evaluate expressions in which letters stand for numbers.

c Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations).

• PEMDAS stands for Parentheses, Exponents, Multiplication, Division, Addition, Subtraction, specifying the order in which operations are performed in interpreting or evaluating numerical expressions.

6.EE.4 Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them).

6.EE.5 Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

and algebraic solutions to simple word problems. For example, how many 44-cent stamps can you buy with \$11? Students are accustomed to solving such problems by division; now they see the parallel with representing the problem algebraically as $0.44n = 11$, from which they use the same reasoning as in the numerical solution to conclude that $n = 11 \div 0.44$.[6.EE.7•] They explore methods such as dividing both sides by the same non-zero number. As word problems grow more complex in Grades 6 and 7, analogous arithmetical and algebraic solutions show the connection between the procedures of solving equations and the reasoning behind those procedures.

When students start studying equations in one variable, it is important for them to understand every occurrence of of a given variable has the same value in the expression and throughout a solution procedure: if x is assumed to be the number satisfying the equation $4x + 3x = 3x + 20$ at the beginning of a solution procedure, it remains that number throughout.

As with all their work with variables, it is important for students to state precisely the meaning of variables they use when setting up equations (MP6). This includes specifying whether the variable refers to a specific number, or to all numbers in some range. For example, in the equation $0.44n = 11$ the variable n refers to a specific number (the number of stamps you can buy for \$11); however, if the expression $0.44n$ is presented as a general formula for calculating the price in dollars of n stamps, then n refers to all numbers in some domain.[6.EE.6] That domain might be specified by inequalities, such as $n > 0$.[6.EE.8]

Represent and analyze quantitative relationships between dependent and independent variables In addition to constructing and solving equations in one variable, students use equations in two variables to express relationships between two quantities that vary together. When they construct an expression like $10 - p$ to represent a quantity such as on page 4, students can choose a variable such as C to represent the calculated quantity and write $C = 10 - p$ to represent the relationship. This prepares students for work with functions in later grades.[6.EE.9] The variable p is the natural choice for the independent variable in this situation, with C the dependent variable. In a situation where the price, p, is to be calculated from the change, C, it might be the other way around.

As they work with such equations students begin to develop a dynamic understanding of variables, an appreciation that they can stand for any number from some domain. This use of variables arises when students study expressions such as $0.44n$, discussed earlier, or equations in two variables such as $d = 5 + 5t$ describing relationship between distance in miles, d, and time in hours, t, for a person starting 5 miles from home and walking away at 5 miles per hour. Students can use tabular[•] and graphical[•] representations to develop an appreciation of varying quantities.

6.EE.7 Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and x are all nonnegative rational numbers.

• In Grade 7, where students learn about complex fractions, this problem can be expressed in cents as well as dollars to help students understand equivalences such as

$$\frac{11}{0.44} = \frac{1100}{44}.$$

Analogous arithmetical and algebraic solutions

J. bought three packs of balloons. He opened them and counted 12 balloons. How many balloons are in a pack?

Arithmetical solution
If three packs have twelve balloons, then one pack has $12 \div 3 = 4$ balloons.

Algebraic solution
Defining the variable: Let b be the number of balloons in a pack.
Writing the equation:

$$3b = 12$$

Solving (mirrors the reasoning of the numerical solution):

$$3b = 12 \rightarrow \frac{3b}{3} = \frac{12}{3}$$
$$b = 4.$$

6.EE.6 Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

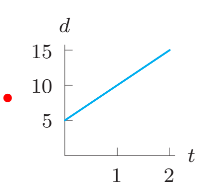
6.EE.8 Write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x > c$ or $x < c$ have infinitely many solutions; represent solutions of such inequalities on number line diagrams.

6.EE.9 Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation.

•

n	1	2	3	4	5
$0.44n$	0.44	0.88	1.32	1.75	2.20

•



Grade 7

Use properties of operations to generate equivalent expressions In Grade 7 students start to simplify general linear expressions• with rational coefficients. Building on work in Grade 6, where students used conventions about the order of operations to parse, and properties of operations to transform, simple expressions such as $2(3+8x)$ or $10-2p$, students now encounter linear expressions with more operations and whose transformation may require an understanding of the rules for multiplying negative numbers, such as $7-2(3-8x)$.[7.EE.1]

• A *general linear expression* in the variable x is a sum of terms which are either rational numbers, or rational numbers times x, e.g., $-\frac{1}{2}+2x+\frac{5}{8}+3x$.

7.EE.1 Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.

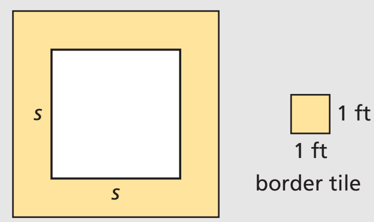
In simplifying this expression students might come up with answers such as

- $5(3-8x)$, mistakenly detaching the 2 from the indicated multiplication
- $7-2(-5x)$, through a determination to perform the computation in parentheses first, even though no simplification is possible
- $7-6-16x$, through an imperfect understanding of the way the distributive law works or of the rules for multiplying negative numbers.

In contrast with the simple linear expressions they see in Grade 6, the more complex expressions students seen in Grade 7 afford shifts of perspective, particularly because of their experience with negative numbers: for example, students might see $7-2(3-8x)$ as $7-(2(3-8x))$ or as $7+(-2)(3+(-8)x)$ (MP7).

As students gain experience with multiple ways of writing an expression, they also learn that different ways of writing expressions can serve different purposes and provide different ways of seeing a problem. For example, $a+0.05a=1.05a$ means that "increase by 5%" is the same as "multiply by 1.05."[7.EE.2] In the example on the right, the connection between the expressions and the figure emphasize that they all represent the same number, and the connection between the structure of each expression and a method of calculation emphasize the fact that expressions are built up from operations on numbers.

Writing expressions in different forms



In expressing the number of tiles needed to border a square pool with side length s feet (where s is a whole number), students might write $4(s+1)$, $s+s+s+s+4$, or $2s+2(s+2)$, each indicating a different way of breaking up the border in order to perform the calculation. They should see all these expressions as equivalent.

7.EE.2 Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related.

Solve real-life and mathematical problems using numerical and algebraic expressions and equations By Grade 7 students start to see whole numbers, integers, and positive and negative fractions as belonging to a single system of rational numbers, and they solve multi-step problems involving rational numbers presented in various forms.[7.EE.3]

7.EE.3 Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. . . .

Students use mental computation and estimation to assess the reasonableness of their solutions. For example, the following statement appeared in an article about the annual migration of the Bar-tailed Godwit from Alaska to New Zealand:

> She had flown for eight days—nonstop—covering approximately 7,250 miles at an average speed of nearly 35 miles per hour.

Students can make the rough mental estimate

$$8 \times 24 \times 35 = 8 \times 12 \times 70 < 100 \times 70 = 7000$$

to recognize that although this astonishing statement is in the right ballpark, the average speed is in fact greater than 35 miles per hour, suggesting that one of the numbers in the article must be wrong.[7.EE.3]

As they build a systematic approach to solving equations in one variable, students continue to compare arithmetical and algebraic solutions to word problems. For example they solve the problem

> The perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?

by subtracting $2 \cdot 6$ from 54 and dividing by 2, and also by setting up the equation

$$2w + 2 \cdot 6 = 54.$$

The steps in solving the equation mirror the steps in the numerical solution. As problems get more complex, algebraic methods become more valuable. For example, in the cyclist problem in the margin, the numerical solution requires some insight in order to keep the cognitive load of the calculations in check. By contrast, choosing the letter s to stand for the unknown speed, students build an equation by adding the distances travelled in three hours ($3s$ and $3 \cdot 12.5$) and setting them equal to 63 to get

$$3s + 3 \cdot 12.5 = 63.$$

It is worthwhile exploring two different possible next steps in the solution of this equation:

$$3s + 37.5 = 64 \quad \text{and} \quad 3(s + 12.5) = 63.$$

The first is suggested by a standard approach to solving linear equations; the second is suggested by a comparison with the numerical solution described earlier.[7.EE.4a]

Students also set up and solve inequalities, recognizing the ways in which the process of solving them is similar to the process of solving linear equations:

> As a salesperson, you are paid \$50 per week plus \$3 per sale. This week you want your pay to be at least \$100. Write an inequality for the number of sales you need to make, and describe the solution.

7.EE.3 . . . Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies.

Looking for structure in word problems (MP7)

> Two cyclists are riding toward each other along a road (each at a constant speed). At 8 am, they are 63 miles apart. They meet at 11 am. If one cyclist rides at 12.5 miles per hour, what is the speed of the other cyclist?

First solution: The first cyclist travels $3 \times 12.5 = 37.5$ miles. The second travels $63 - 37.5 = 25.5$ miles, so goes $\frac{25.5}{3} = 8.5$ miles per hour. Another solution uses a key hidden quantity, the speed at which the cyclists are approaching each other, to simplify the calculations: since $\frac{63}{3} = 21$, the cyclists are approaching each other at 21 miles per hour, so the other cyclist is traveling at $21 - 12.5 = 8.5$ miles per hour.

7.EE.4a Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

a. Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach.

Students also recognize one important new consideration in solving inequalities: multiplying or dividing both sides of an inequality by a negative number reverses the order of the comparison it represents. It is useful to present contexts that allows students to make sense of this. For example,

> If the price of a ticket to a school concert is p dollars then the attendance is $1000 - 50p$. What range of prices ensures that at least 600 people attend?

Students recognize that the requirement of at least 600 people leads to the inequality $1000 - 50p \geqslant 600$. Before solving the inequality, they use common sense to anticipate that that answer will be of the form $p \leqslant ?$, since higher prices result in lower attendance.[7.EE.4b] (Note that inequalities using $\leqslant$ and $\geqslant$ are included in this standard, in addition to $>$ and $<$.)

7.EE.4b Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

b Solve word problems leading to inequalities of the form $px + q > r$ or $px + q < r$, where p, q, and r are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem.

Grade 8

Work with radicals and integer exponents In Grade 8 students add the properties of integer exponents to their repertoire of rules for transforming expressions.• Students have been denoting whole number powers of 10 with exponential notation since Grade 5, and they have seen the pattern in the number of zeros when powers of 10 are multiplied. They express this as $10^a 10^b = 10^{a+b}$ for whole numbers a and b. Requiring this rule to hold when a and b are integers leads to the definition of the meaning of powers with 0 and negative exponents. For example, we define $10^0 = 1$ because we want $10^a 10^0 = 10^{a+0} = 10^a$, so 10^0 must equal 1. Students extend these rules to other bases, and learn other properties of exponents.[8.EE.1]

Notice that students do not learn the properties of rational exponents until high school. However, they prepare in Grade 8 by starting to work systematically with the square root and cube root symbols, writing, for example, $\sqrt{64} = \sqrt{8^2} = 8$ and $(\sqrt[3]{5})^3 = 5$. Since $\sqrt{p}$ is defined to mean the positive solution to the equation $x^2 = p$ (when it exists), it is not correct to say (as is common) that $\sqrt{64} = \pm 8$. On the other hand, in describing the solutions to $x^2 = 64$, students can write $x = \pm\sqrt{64} = \pm 8$.[8.EE.2] Students in Grade 8 are not in a position to prove that these are the only solutions, but rather use informal methods such as guess and check.

Students gain experience with the properties of exponents by working with estimates of very large and very small quantities. For example, they estimate the population of the United States as 3×10^8 and the population of the world as 7×10^9, and determine that the world population is more than 20 times larger.[8.EE.3] They express and perform calculations with very large numbers using scientific notation. For example, given that we breathe about 6 liters of air per minute, they estimate that there are $60 \times 24 = 6 \times 2.4 \times 10^2 \approx 1.5 \times 10^3$ minutes in a day, and that we therefore breath about $6 \times 1.5 \times 10^3 \approx 10^4$ liters in a day. In a lifetime of 75 years there are about $365 \times 75 \approx 3 \times 10^4$ days, and so we breath about $3 \times 10^4 \times 10^4 = 3 \times 10^8$ liters of air in a lifetime.[8.EE.4]

• **Properties of Integer Exponents**
For any nonzero rational numbers a and b and integers n and m:

1. $a^n a^m = a^{n+m}$
2. $(a^n)^m = a^{nm}$
3. $a^n b^n = (ab)^n$
4. $a^0 = 1$
5. $a^{-n} = 1/a^n$

8.EE.1 Know and apply the properties of integer exponents to generate equivalent numerical expressions.

8.EE.2 Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

8.EE.3 Use numbers expressed in the form of a single digit times a whole-number power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other.

8.EE.4 Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology.

Understand the connections between proportional relationships, lines, and linear equations As students in Grade 8 move towards an understanding of the idea of a function, they begin to tie together a number of notions that have been developing over the last few grades:

1. An expression in one variable defines a general calculation in which the variable can represent a range of numbers—an input-output machine with the variable representing the input and the expression calculating the output. For example, $60t$ is the distance traveled in t hours by a car traveling at a constant speed of 60 miles per hour.

2. Choosing a variable to represent the output leads to an equation in two variables describing the relation between two quantities. For example, choosing d to represent the distance traveled by the car traveling at 65 miles per hour yields the equation $d = 65t$. Reading the expression on the right (multiplication of the variable by a constant) reveals the relationship (a rate relationship in which distance is proportional to time).

3. Tabulating values of the expression is the same as tabulating solution pairs of the corresponding equation.• This gives insight into the nature of the relationship; for example, that the distance increases by the same amount for the same increase in the time (the ratio between the two being the speed).

4. Plotting points on the coordinate plane, in which each axis is marked with a scale representing one quantity, affords a visual representation of the relationship between two quantities.•

Proportional relationships provide a fruitful first ground in which these notions can grow together. The constant of proportionality is visible in each; as the multiplicative factor in the expression, as the slope of the line, and as an increment in the table (if the dependent variable goes up by 1 unit in each entry).• As students start to build a unified notion of the concept of function they are able to compare proportional relationships presented in different ways. For example, the table shows 300 miles in 5 hours, whereas the graph shows more than 300 miles in the same time.[8.EE.5]

The connection between the unit rate in a proportional relationship and the slope of its graph depends on a connection with the geometry of similar triangles. The fact that a line has a well-defined slope—that the ratio between the rise and run for any two points on the line is always the same—depends on similar triangles.[8.EE.6]

The fact that the slope is constant between any two points on a line leads to the derivation of an equation for the line. For a line through the origin, the right triangle whose hypotenuse is the line segment from $(0, 0)$ to a point (x, y) on the line is similar to the right triangle from $(0, 0)$ to the point $(1, m)$ on the line, and so

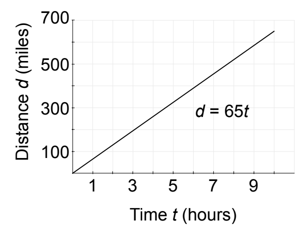
$$\frac{y}{x} = \frac{m}{1}, \quad \text{or} \quad y = mx.$$

The equation for a line not through the origin can be derived in a similar way, starting from the y-intercept $(0, b)$ instead of the origin.

Analyze and solve linear equations and pairs of simultaneous linear equations By Grade 8 students have the tools to solve an equation which has a general linear expression on each side of the equal sign,[8.EE.7] for example:

> If a bar of soap balances $\frac{3}{4}$ of a bar of soap and $\frac{3}{4}$ of a pound, how much does the bar of soap weigh?

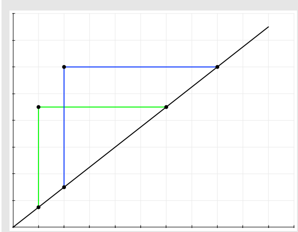
t (hours)	1	2	3	4	5	6
$60t$ (miles)	60	120	180	240	300	360



• In the Grade 8 Functions domain, students see the relationship between the graph of a proportional relationship and its equation $y = mx$ as a special case of the relationship between a line and its equation $y = mx + b$, with $b = 0$.

8.EE.5 Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways.

Why lines have constant slope



The green triangle is similar to the blue triangle because corresponding angles are equal, so the ratio of rise to run is the same in each.

8.EE.6 Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

8.EE.7 Solve linear equations in one variable.

This is an example where choosing a letter, say b, to represent the weight of the bar of soap and solving the equation

$$b = \frac{3}{4}b + \frac{3}{4}$$

is probably easier for students than reasoning through a numerical solution. Linear equations also arise in problems where two linear functions are compared. For example

> Henry and Jose are gaining weight for football. Henry weighs 205 pounds and is gaining 2 pounds per week. Jose weighs 195 pounds and is gaining 3 pounds per week. When will they weigh the same?

Students in Grade 8 also start to solve problems that lead to simultaneous equations,[8.EE.8] for example

> Tickets for the class show are \$3 for students and \$10 for adults. The auditorium holds 450 people. The show was sold out and the class raised \$2750 in ticket sales. How many students bought tickets?

This problem involves two variables, the number S of student tickets sold and the number A of adult tickets sold, and imposes two constraints on those variables: the number of tickets sold is 450 and the dollar value of tickets sold is 2750.

[8.EE.8] Analyze and solve pairs of simultaneous linear equations.

Progressions for the Common Core State Standards in Mathematics (draft)

29 May 2011

K, Counting and Cardinality; K–5, Operations and Algebraic Thinking

Counting and Cardinality and Operations and Algebraic Thinking are about understanding and using numbers. Counting and Cardinality underlies Operations and Algebraic Thinking as well as Number and Operations in Base Ten. It begins with early counting and telling how many in one group of objects. Addition, subtraction, multiplication, and division grow from these early roots. From its very beginnings, this Progression involves important ideas that are neither trivial nor obvious; these ideas need to be taught, in ways that are interesting and engaging to young students.

The Progression in Operations and Algebraic Thinking deals with the basic operations—the kinds of quantitative relationships they model and consequently the kinds of problems they can be used to solve as well as their mathematical properties and relationships. Although most of the standards organized under the OA heading involve whole numbers, the importance of the Progression is much more general because it describes concepts, properties, and representations that extend to other number systems, to measures, and to algebra. For example, if the mass of the sun is x kilograms, and the mass of the rest of the solar system is y kilograms, then the mass of the solar system as a whole is the sum $x + y$ kilograms. In this example of additive reasoning, it doesn't matter whether x and y are whole numbers, fractions, decimals, or even variables. Likewise, a property such as distributivity holds for all the number systems that students will study in K–12, including complex numbers.

The generality of the concepts involved in Operations and Algebraic Thinking means that students' work in this area should be designed to help them extend arithmetic beyond whole numbers (see the NF and NBT Progressions) and understand and apply expressions and equations in later grades (see the EE Progression).

Addition and subtraction are the first operations studied. Initially, the meaning of addition is separate from the meaning of subtraction, and students build relationships between addition and subtraction over time. Subtraction comes to be understood as reversing the actions involved in addition and as finding an unknown addend. Likewise, the meaning of multiplication is initially separate from the meaning of division, and students gradually perceive relationships between division and multiplication analogous to those between addition and subtraction, understanding division as reversing the actions involved in multiplication and finding an unknown product.

Over time, students build their understanding of the properties of arithmetic: commutativity and associativity of addition and multiplication, and distributivity of multiplication over addition. Initially, they build intuitive understandings of these properties, and they use these intuitive understandings in strategies to solve real-world and mathematical problems. Later, these understandings become more explicit and allow students to extend operations into the system of rational numbers.

As the meanings and properties of operations develop, students develop computational methods in tandem. The OA Progression in Kindergarten and Grade 1 describes this development for single-digit addition and subtraction, culminating in methods that rely on properties of operations. The NBT Progression describes how these methods combine with place value reasoning to extend computation to multi-digit numbers. The NF Progression describes how the meanings of operations combine with fraction concepts to extend computation to fractions.

Students engage in the Standards for Mathematical Practice in grade-appropriate ways from Kindergarten to Grade 5. Pervasive classroom use of these mathematical practices in each grade affords students opportunities to develop understanding of operations and algebraic thinking.

Counting and Cardinality

Several progressions originate in knowing number names and the count sequence:K.CC.1

K.CC.1 Count to 100 by ones and by tens.

From saying the counting words to counting out objects Students usually know or can learn to say the counting words up to a given number before they can use these numbers to count objects or to tell the number of objects. Students become fluent in saying the count sequence so that they have enough attention to focus on the pairings involved in counting objects. To count a group of objects, they pair each word said with one object.K.CC.4a This is usually facilitated by an indicating act (such as pointing to objects or moving them) that keeps each word said in time paired to one and only one object located in space. Counting objects arranged in a line is easiest; with more practice, students learn to count objects in more difficult arrangements, such as rectangular arrays (they need to ensure they reach every row or column and do not repeat rows or columns); circles (they need to stop just before the object they started with); and scattered configurations (they need to make a single path through all of the objects).K.CC.5 Later, students can count out a given number of objects,K.CC.5 which is more difficult than just counting that many objects, because counting must be fluent enough for the student to have enough attention to remember the number of objects that is being counted out.

K.CC.4a Understand the relationship between numbers and quantities; connect counting to cardinality.

a When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.

K.CC.5 Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

From subitizing to single-digit arithmetic fluency Students come to quickly recognize the cardinalities of small groups without having to count the objects; this is called *perceptual subitizing*. Perceptual subitizing develops into *conceptual subitizing*—recognizing that a collection of objects is composed of two subcollections and quickly combining their cardinalities to find the cardinality of the collection (e.g., seeing a set as two subsets of cardinality 2 and saying "four"). Use of conceptual subitizing in adding and subtracting small numbers progresses to supporting steps of more advanced methods for adding, subtracting, multiplying, and dividing single-digit numbers (in several OA standards from Grade 1 to 3 that culminate in single-digit fluency).

From counting to counting on Students understand that the last number name said in counting tells the number of objects counted.K.CC.4b Prior to reaching this understanding, a student who is asked "How many kittens?" may regard the counting performance itself as the answer, instead of answering with the cardinality of the set. Experience with counting allows students to discuss and come to understand the second part of K.CC.4b—that the number of objects is the same regardless of their arrangement or the order in which they were counted. This connection will continue in Grade 1 with the

K.CC.4b Understand the relationship between numbers and quantities; connect counting to cardinality.

b Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.

more advanced counting-on methods in which a counting word represents a group of objects that are added or subtracted and addends become embedded within the total[1.OA.6] (see page 14). Being able to count forward, beginning from a given number within the known sequence,[K.CC.2] is a prerequisite for such counting on. Finally, understanding that each successive number name refers to a quantity that is one larger[K.CC.4c] is the conceptual start for Grade 1 counting on. Prior to reaching this understanding, a student might have to recount entirely a collection of known cardinality to which a single object has been added.

1.OA.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$); decomposing a number leading to a ten (e.g., $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$); using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$).

K.CC.2 Count forward beginning from a given number within the known sequence (instead of having to begin at 1).

K.CC.4c Understand the relationship between numbers and quantities; connect counting to cardinality.

c Understand that each successive number name refers to a quantity that is one larger.

From spoken number words to written base-ten numerals to base-ten system understanding The NBT Progression discusses the special role of 10 and the difficulties that English speakers face because the base-ten structure is not evident in all the English number words.

From comparison by matching to comparison by numbers to comparison involving adding and subtracting The standards about comparing numbers[K.CC.6,K.CC.7] focus on students identifying which of two groups has more than (or fewer than, or the same amount as) the other. Students first learn to match the objects in the two groups to see if there are any extra and then to count the objects in each group and use their knowledge of the count sequence to decide which number is greater than the other (the number farther along in the count sequence). Students learn that even if one group looks as if it has more objects (e.g., has some extra sticking out), matching or counting may reveal a different result. Comparing numbers progresses in Grade 1 to adding and subtracting in comparing situations (finding out "how many more" or "how many less"[1.OA.1] and not just "which is more" or "which is less").

K.CC.6 Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.

K.CC.7 Compare two numbers between 1 and 10 presented as written numerals.

1.OA.1 Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

Operations and Algebraic Thinking

Overview of Grades K–2

Students develop meanings for addition and subtraction as they encounter problem situations in Kindergarten, and they extend these meanings as they encounter increasingly difficult problem situations in Grade 1. They represent these problems in increasingly sophisticated ways. And they learn and use increasingly sophisticated computation methods to find answers. In each grade, the situations, representations, and methods are calibrated to be coherent and to foster growth from one grade to the next.

The main addition and subtraction situations students work with are listed in Table 1. The computation methods they learn to use are summarized in the margin and described in more detail in the Appendix.

Methods used for solving single-digit addition and subtraction problems

Level 1. Direct Modeling by Counting All or Taking Away. Represent situation or numerical problem with groups of objects, a drawing, or fingers. Model the situation by composing two addend groups or decomposing a total group. Count the resulting total or addend.

Level 2. Counting On. Embed an addend within the total (the addend is perceived simultaneously as an addend and as part of the total). Count this total but abbreviate the counting by omitting the count of this addend; instead, begin with the number word of this addend. Some method of keeping track (fingers, objects, mentally imaged objects, body motions, other count words) is used to monitor the count.

For addition, the count is stopped when the amount of the remaining addend has been counted. The last number word is the total. For subtraction, the count is stopped when the total occurs in the count. The tracking method indicates the difference (seen as an unknown addend).

Level 3. Convert to an Easier Problem. Decompose an addend and compose a part with another addend.

See Appendix for examples and further details.

Table 1: Addition and subtraction situations

	Result Unknown	Change Unknown	Start Unknown
Add To	A bunnies sat on the grass. B more bunnies hopped there. How many bunnies are on the grass now? $A + B = \square$	A bunnies were sitting on the grass. Some more bunnies hopped there. Then there were C bunnies. How many bunnies hopped over to the first A bunnies? $A + \square = C$	Some bunnies were sitting on the grass. B more bunnies hopped there. Then there were C bunnies. How many bunnies were on the grass before? $\square + B = C$
Take From	C apples were on the table. I ate B apples. How many apples are on the table now? $C - B = \square$	C apples were on the table. I ate some apples. Then there were A apples. How many apples did I eat? $C - \square = A$	Some apples were on the table. I ate B apples. Then there were A apples. How many apples were on the table before? $\square - B = A$
	Total Unknown	**Both Addends Unknown[1]**	**Addend Unknown[2]**
Put Together /Take Apart	A red apples and B green apples are on the table. How many apples are on the table? $A + B = \square$	Grandma has C flowers. How many can she put in her red vase and how many in her blue vase? $C = \square + \square$	C apples are on the table. A are red and the rest are green. How many apples are green? $A + \square = C$ $C - A = \square$
	Difference Unknown	**Bigger Unknown**	**Smaller Unknown**
Compare	*"How many more?" version.* Lucy has A apples. Julie has C apples. How many more apples does Julie have than Lucy? *"How many fewer?" version.* Lucy has A apples. Julie has C apples. How many fewer apples does Lucy have than Julie? $A + \square = C$ $C - A = \square$	*"More" version suggests operation.* Julie has B more apples than Lucy. Lucy has A apples. How many apples does Julie have? *"Fewer" version suggests wrong operation.* Lucy has B fewer apples than Julie. Lucy has A apples. How many apples does Julie have? $A + B = \square$	*"Fewer" version suggests operation.* Lucy has B fewer apples than Julie. Julie has C apples. How many apples does Lucy have? *"More" suggests wrong operation.* Julie has B more apples than Lucy. Julie has C apples. How many apples does Lucy have? $C - B = \square$ $\square + B = C$

In each type (shown as a row), any one of the three quantities in the situation can be unknown, leading to the subtypes shown in each cell of the table. The table also shows some important language variants which, while mathematically the same, require separate attention. Other descriptions of the situations may use somewhat different names. Adapted from CCSS, p. 88, which is based on *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, National Research Council, 2009, pp. 32–33.

[1] This can be used to show all decompositions of a given number, especially important for numbers within 10. Equations with totals on the left help children understand that = does not always mean "makes" or "results in" but always means "is the same number as." Such problems are not a problem subtype with one unknown, as is the Addend Unknown subtype to the right. These problems are a productive variation with two unknowns that give experience with finding all of the decompositions of a number and reflecting on the patterns involved.

[2] Either addend can be unknown; both variations should be included.

Kindergarten

Students act out adding and subtracting situations by representing quantities in the situation with objects, their fingers, and math drawings (MP5).K.OA.1 To do this, students must mathematize a real-world situation (MP4), focusing on the quantities and their relationships rather than non-mathematical aspects of the situation. Situations can be acted out and/or presented with pictures or words. Math drawings facilitate reflection and discussion because they remain after the problem is solved. These concrete methods that show all of the objects are called Level 1 methods.

K.OA.1 Represent addition and subtraction with objects, fingers, mental images, drawings, sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.

Students learn and use mathematical and non-mathematical language, especially when they make up problems and explain their representation and solution. The teacher can write expressions (e.g., $3-1$) to represent operations, as well as writing equations that represent the whole situation before the solution (e.g., $3-1=\square$) or after (e.g., $3-1=2$). Expressions like $3-1$ or $2+1$ show the operation, and it is helpful for students to have experience just with the expression so they can conceptually chunk this part of an equation.

Working within 5 Students work with small numbers first, though many kindergarteners will enter school having learned parts of the Kindergarten standards at home or at a preschool program. Focusing attention on small groups in adding and subtracting situations can help students move from perceptual subitizing to conceptual subitizing in which they see and say the addends and the total,• e.g., "Two and one make three."

• Note on vocabulary: The term "total" is used here instead of the term "sum." "Sum" sounds the same as "some," but has the opposite meaning. "Some" is used to describe problem situations with one or both addends unknown, so it is better in the earlier grades to use "total" rather than "sum." Formal vocabulary for subtraction ("minuend" and "subtrahend") is not needed for Kindergarten, Grade 1, and Grade 2, and may inhibit students seeing and discussing relationships between addition and subtraction. At these grades, the terms "total" and "addend" are sufficient for classroom discussion.

Students will generally use fingers for keeping track of addends and parts of addends for the Level 2 and 3 methods used in later grades, so it is important that students in Kindergarten develop rapid visual and kinesthetic recognition of numbers to 5 on their fingers. Students may bring from home different ways to show numbers with their fingers and to raise (or lower) them when counting. The three major ways used around the world are starting with the thumb, the little finger, or the pointing finger (ending with the thumb in the latter two cases). Each way has advantages physically or mathematically, so students can use whatever is familiar to them. The teacher can use the range of methods present in the classroom, and these methods can be compared by students to expand their understanding of numbers. Using fingers is not a concern unless it remains at the first level of direct modeling in later grades.

Students in Kindergarten work with the following types of addition and subtraction situations: Add To with Result Unknown; Take From with Result Unknown; and Put Together/Take Apart with Total Unknown and Both Addends Unknown (see the dark shaded types in Table 2). Add To/Take From situations are action-oriented; they show changes from an initial state to a final state. These situations are readily modeled by equations because each aspect of the situation has a representation as number, operation (+ or −), or equal sign (=, here with the meaning of "becomes," rather than the more general "equals").

Table 2: Addition and subtraction situations by grade level.

	Result Unknown	Change Unknown	Start Unknown
Add To	A bunnies sat on the grass. B more bunnies hopped there. How many bunnies are on the grass now? $A + B = \square$	A bunnies were sitting on the grass. Some more bunnies hopped there. Then there were C bunnies. How many bunnies hopped over to the first A bunnies? $A + \square = C$	Some bunnies were sitting on the grass. B more bunnies hopped there. Then there were C bunnies. How many bunnies were on the grass before? $\square + B = C$
Take From	C apples were on the table. I ate B apples. How many apples are on the table now? $C - B = \square$	C apples were on the table. I ate some apples. Then there were A apples. How many apples did I eat? $C - \square = A$	Some apples were on the table. I ate B apples. Then there were A apples. How many apples were on the table before? $\square - B = A$
	Total Unknown	**Both Addends Unknown[1]**	**Addend Unknown[2]**
Put Together /Take Apart	A red apples and B green apples are on the table. How many apples are on the table? $A + B = \square$	Grandma has C flowers. How many can she put in her red vase and how many in her blue vase? $C = \square + \square$	C apples are on the table. A are red and the rest are green. How many apples are green? $A + \square = C$ $C - A = \square$
	Difference Unknown	**Bigger Unknown**	**Smaller Unknown**
Compare	*"How many more?" version.* Lucy has A apples. Julie has C apples. How many more apples does Julie have than Lucy? *"How many fewer?" version.* Lucy has A apples. Julie has C apples. How many fewer apples does Lucy have than Julie? $A + \square = C$ $C - A = \square$	*"More" version suggests operation.* Julie has B more apples than Lucy. Lucy has A apples. How many apples does Julie have? *"Fewer" version suggests wrong operation.* Lucy has B fewer apples than Julie. Lucy has A apples. How many apples does Julie have? $A + B = \square$	*"Fewer" version suggests operation.* Lucy has B fewer apples than Julie. Julie has C apples. How many apples does Lucy have? *"More" version suggests wrong operation.* Julie has B more apples than Lucy. Julie has C apples. How many apples does Lucy have? $C - B = \square$ $\square + B = C$

Darker shading indicates the four Kindergarten problem subtypes. Grade 1 and 2 students work with all subtypes and variants. Unshaded (white) problems are the four difficult subtypes or variants that students should work with in Grade 1 but need not master until Grade 2. Adapted from CCSS, p. 88, which is based on *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, National Research Council, 2009, pp. 32–33.

[1] This can be used to show all decompositions of a given number, especially important for numbers within 10. Equations with totals on the left help children understand that = does not always mean "makes" or "results in" but always means "is the same number as." Such problems are not a problem subtype with one unknown, as is the Addend Unknown subtype to the right. These problems are a productive variation with two unknowns that give experience with finding all of the decompositions of a number and reflecting on the patterns involved.

[2] Either addend can be unknown; both variations should be included.

In Put Together/Take Apart situations, two quantities jointly compose a third quantity (the total), or a quantity can be decomposed into two quantities (the addends). This composition/decomposition may be physical or conceptual. These situations are acted out with objects initially and later children begin to move to conceptual mental actions of shifting between seeing the addends and seeing the total (e.g., seeing children or seeing boys and girls, or seeing red and green apples or all the apples).

The relationship between addition and subtraction in the Add To/Take From and the Put Together/Take Apart action situations is that of reversibility of actions: an Add To situation undoes a Take From situation and vice versa and a composition (Put Together) undoes a decomposition (Take Apart) and vice versa.

Put Together/Take Apart situations with Both Addends Unknown play an important role in Kindergarten because they allow students to explore various compositions that make each number.[K.OA.3] This will help students to build the Level 2 embedded number representations used to solve more advanced problem subtypes. As students decompose a given number to find all of the partners• that compose the number, the teacher can record each decomposition with an equation such as $5 = 4 + 1$, showing the total on the left and the two addends on the right.• Students can find patterns in all of the decompositions of a given number and eventually summarize these patterns for several numbers.

Equations with one number on the left and an operation on the right (e.g., $5 = 2 + 3$ to record a group of 5 things decomposed as a group of 2 things and a group of 3 things) allow students to understand equations as showing in various ways that the quantities on both sides have the same value.[MP6]

K.OA.3 Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., $5 = 2+3$ and $5 = 4 + 1$).

• The two addends that make a total can also be called partners in Kindergarten and Grade 1 to help children understand that they are the two numbers that go together to make the total.

• For each total, two equations involving 0 can be written, e.g., $5 = 5 + 0$ and $5 = 0 + 5$. Once students are aware that such equations can be written, practice in decomposing is best done without such 0 cases.

MP6 Working toward "using the equal sign consistently and appropriately."

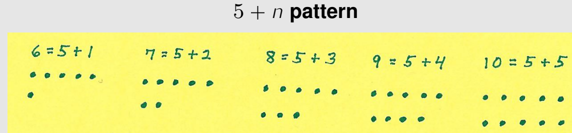
Working within 10 Students expand their work in addition and subtraction from within 5 to within 10. They use the Level 1 methods developed for smaller totals as they represent and solve problems with objects, their fingers, and math drawings. Patterns such as "adding one is just the next counting word"[K.CC.4c] and "adding zero gives the same number" become more visible and useful for all of the numbers from 1 to 9. Patterns such as the $5 + n$ pattern used widely around the world play an important role in learning particular additions and subtractions, and later as patterns in steps in the Level 2 and 3 methods. Fingers can be used to show the same 5-patterns, but students should be asked to explain these relationships explicitly because they may not be obvious to all students.[MP3] As the school year progresses, students internalize their external representations and solution actions, and mental images become important in problem representation and solution.

Student drawings show the relationships in addition and subtraction situations for larger numbers (6 to 9) in various ways, such

K.CC.4c Understand the relationship between numbers and quantities; connect counting to cardinality.

c Understand that each successive number name refers to a quantity that is one larger.

$5 + n$ **pattern**



MP3 Students explain their conclusions to others.

as groupings, things crossed out, numbers labeling parts or totals, and letters or words labeling aspects of the situation. The symbols $+$, $-$, or $=$ may be in the drawing. Students should be encouraged to explain their drawings and discuss how different drawings are the same and different.[MP1]

Later in the year, students solve addition and subtraction equations for numbers within 5, for example, $2 + 1 = \square$ or $3 - 1 = \square$, while still connecting these equations to situations verbally or with drawings. Experience with decompositions of numbers and with Add To and Take From situations enables students to begin to fluently add and subtract within 5.[K.OA.5]

Finally, composing and decomposing numbers from 11 to 19 into ten ones and some further ones builds from all this work.[K.NBT.1] This is a vital first step kindergarteners must take toward understanding base-ten notation for numbers greater than 9. (See the NBT Progression.)

The Kindergarten standards can be stated succinctly, but they represent a great deal of focused and rich interactions in the classroom. This is necessary in order to enable all students to understand all of the numbers and concepts involved. Students who enter Kindergarten without knowledge of small numbers or of counting to ten will require extra teaching time in Kindergarten to meet the standards. Such time and support are vital for enabling all students to master the Grade 1 standards in Grade 1.

MP1 Understand the approaches of others and identify correspondences

K.OA.5 Fluently add and subtract within 5.

K.NBT.1 Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

Grade 1

Students extend their work in three major and interrelated ways, by:

- Representing and solving a new type of problem situation (Compare);
- Representing and solving the subtypes for all unknowns in all three types;
- Using Level 2 and Level 3 methods to extend addition and subtraction problem solving beyond 10, to problems within 20. In particular, the OA progression in Grade 1 deals with adding two single-digit addends, and related subtractions.•

Representing and solving a new type of problem situation (Compare) In a Compare situation, two quantities are compared to find "How many more" or "How many less."•K.CC.6,K.CC.7 One reason Compare problems are more advanced than the other two major types is that in Compare problems, one of the quantities (the difference) is not present in the situation physically, and must be conceptualized and constructed in a representation, by showing the "extra" that when added to the smaller unknown makes the total equal to the bigger unknown or by finding this quantity embedded within the bigger unknown.

The language of comparisons is also difficult. For example, "Julie has three more apples than Lucy" tells both that Julie has more apples and that the difference is three. Many students "hear" the part of the sentence about who has more, but do not initially hear the part about how many more; they need experience hearing and saying a separate sentence for each of the two parts in order to comprehend and say the one-sentence form. Another language issue is that the comparing sentence might be stated in either of two related ways, using "more" or "less." Students need considerable experience with "less" to differentiate it from "more"; some children think that "less" means "more." Finally, as well as the basic "How many more/less" question form, the comparing sentence might take an active, equalizing and counterfactual form (e.g., "How many more apples does Lucy need to have as many as Julie?") or might be stated in a static and factual way as a question about how many things are unmatched (e.g., "If there are 8 trucks and 5 drivers, how many trucks do not have a driver?"). Extensive experience with a variety of contexts is needed to master these linguistic and situational complexities. Matching with objects and with drawings, and labeling each quantity (e.g., J or Julie and L or Lucy) is helpful. Later in Grade 1, a tape diagram can be used. These comparing diagrams can continue to be used for multi-digit numbers, fractions, decimals, and variables, thus connecting understandings of these numbers in

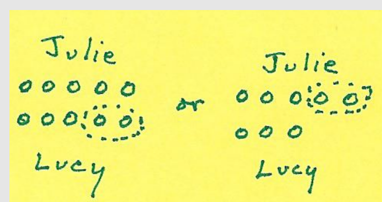
• Other Grade 1 problems within 20, such as $14+5$, are best viewed in the context of place value, i.e., associated with 1.NBT.4. See the NBT Progression.

• Compare problems build upon Kindergarten comparisons, in which students identified "Which is more?" or "Which is less?" without ascertaining the difference between the numbers.

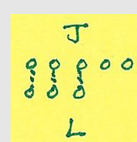
K.CC.6 Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.

K.CC.7 Compare two numbers between 1 and 10 presented as written numerals.

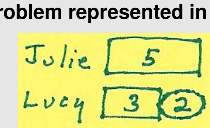
Representing the difference in a Compare problem



Compare problem solved by matching



Compare problem represented in tape diagram



comparing situations with such situations for single-digit numbers. The labels can get more detailed in later grades.

Some textbooks represent all Compare problems with a subtraction equation, but that is not how many students think of the subtypes. Students represent Compare situations in different ways, often as an unknown addend problem (see Table 1). If textbooks and teachers model representations of or solution methods for Compare problems, these should reflect the variability students show. In all mathematical problem solving, what matters is the explanation a student gives to relate a representation to a context, and not the representation separated from its context.

Representing and solving the subtypes for all unknowns in all three types In Grade 1, students solve problems of all twelve subtypes (see Table 2) including both language variants of Compare problems. Initially, the numbers in such problems are small enough that students can make math drawings showing all the objects in order to solve the problem. Students then represent problems with equations, called situation equations. For example, a situation equation for a Take From problem with Result Unknown might read $14 - 8 = \square$.

Put Together/Take Apart problems with Addend Unknown afford students the opportunity to see subtraction as the opposite of addition in a different way than as reversing the action, namely as finding an unknown addend.[1.OA.4] The meaning of subtraction as an unknown-addend addition problem is one of the essential understandings students will need in middle school in order to extend arithmetic to negative rational numbers.

[1.OA.4] Understand subtraction as an unknown-addend problem.

Students next gain experience with the more difficult and more "algebraic" problem subtypes in which a situation equation does not immediately lead to the answer. For example, a student analyzing a Take From problem with Change Unknown might write the situation equation $14 - \square = 8$. This equation does not immediately lead to the answer. To make progress, the student can write a related equation called a solution equation—in this case, either $8 + \square = 14$ or $14 - 8 = \square$. These equations both lead to the answer by Level 2 or Level 3 strategies (see discussion in the next section).

Students thus begin developing an algebraic perspective many years before they will use formal algebraic symbols and methods. They read to understand the problem situation, represent the situation and its quantitative relationships with expressions and equations, and then manipulate that representation if necessary, using properties of operations and/or relationships between operations. Linking equations to concrete materials, drawings, and other representations of problem situations affords deep and flexible understandings of these building blocks of algebra. Learning where the total is in addition equations (alone on one side of the equal sign) and in subtraction equations (to the left of the minus sign) helps students move from a situation equation to a related solution equation.

Because the language and conceptual demands are high, some students in Grade 1 may not master the most difficult subtypes of word problems, such as Compare problems that use language opposite to the operation required for solving (see the unshaded subtypes and variants in Table 2). Some students may also still have difficulty with the conceptual demands of Start Unknown problems. Grade 1 children should have an opportunity to solve and discuss such problems, but proficiency on grade level tests with these most difficult subtypes should wait until Grade 2 along with the other extensions of problem solving.

Using Level 2 and Level 3 strategies to extend addition and subtraction problem solving beyond 10, to problems within 20 As Grade 1 students are extending the range of problem types and subtypes they can solve, they are also extending the range of numbers they deal with[1.OA.6] and the sophistication of the methods they use to add and subtract within this larger range.[1.OA.1,1.OA.8]

The advance from Level 1 methods to Level 2 methods can be clearly seen in the context of situations with unknown addends.[1] These are the situations that can be represented by an addition equation with one unknown addend, e.g., $9 + \square = 13$. Students can solve some unknown addend problems by trial and error or by knowing the relevant decomposition of the total. But a Level 2 counting on solution involves seeing the 9 as part of 13, and understanding that counting the 9 things can be "taken as done" if we begin the count from 9: thus the student may say,

"Niiiiine,	ten,	eleven,	twelve,	thirteen."
	1	2	3	4

Students keep track of how many they counted on (here, 4) with fingers, mental images, or physical actions such as head bobs. Elongating the first counting word ("Niiiiine...") is natural and indicates that the student differentiates between the first addend and the counts for the second addend. Counting on enables students to add and subtract easily within 20 because they do not have to use fingers to show totals of more than 10 which is difficult. Students might also use the commutative property to shorten tasks, by counting on from the larger addend even if it is second (e.g., for $4 + 9$, counting on from 9 instead of from 4).

Counting on should be seen as a thinking strategy, not a rote method. It involves seeing the first addend as embedded in the total, and it involves a conceptual interplay between counting and the cardinality in the first addend (shifting from the cardinal meaning of the first addend to the counting meaning). Finally, there is a level of abstraction involved in counting on, because students are counting

1.OA.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$); decomposing a number leading to a ten (e.g., $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$); using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$).

1.OA.1 Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

1.OA.8 Determine the unknown whole number in an addition or subtraction equation relating three whole numbers.

[1] Grade 1 students also solve the easy Kindergarten problem subtypes by counting on.

the words rather than objects. Number words have become objects to students.

Counting on can be used to add (find a total) or subtract (find an unknown addend). To an observer watching the student, adding and subtracting look the same. Whether the problem is $9+4$ or $13-9$, we will hear the student say the same thing: "Niiiiine, ten, eleven, twelve, thirteen" with four head bobs or four fingers unfolding. The differences are in what is being monitored to know when to stop, and what gives the answer.

Counting on to add and subtract

$9+4$

"Niiiiine, ten, eleven, twelve, thirteen."

1 2 3 4

$13-9$

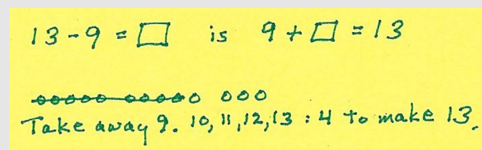
"Niiiiine, ten, eleven, twelve, thirteen."

1 2 3 4

When counting on to add $9+4$, the student is counting the fingers or head bobs to know when to stop counting aloud, and the last counting word said gives the answer. For counting on to subtract $13-9$, the opposite is true: the student is listening to counting words to know when to stop, and the accumulated fingers or head bobs give the answer.

Students in many countries learn counting forward methods of subtracting, including counting on. Counting on for subtraction is easier than counting down. Also, unlike counting down, counting on reinforces that subtraction is an unknown-addend problem. Learning to think of and solve subtractions as unknown addend problems makes subtraction as easy as addition (or even easier), and it emphasizes the relationship between addition and subtraction. The taking away meaning of subtraction can be emphasized within counting on by showing the total and then taking away the objects that are at the *beginning*. In a drawing this taking away can be shown with a horizontal line segment suggesting a minus sign. So one can think of the $9+\square=13$ situation as "I took away 9. I now have 10, 11, 12, 13 [stop when I hear 13], so 4 are left because I counted on 4 from 9 to get to 13." Taking away objects at the end suggests counting down, which is more difficult than counting on. Showing 13 decomposed in groups of five as in the illustration to the right also supports students seeing how to use the Level 3 make-a-ten method; 9 needs 1 more to make 10 and there are 3 more in 13, so 4 from 9 to 13.

"Taking away" indicated with horizontal line segment and solving by counting on to 13



Level 3 methods involve decomposing an addend and composing it with the other addend to form an equivalent but easier problem. This relies on properties of operations.[1.OA.3] Students do not necessarily have to justify their representations or solution using properties, but they can begin to learn to recognize these properties in action and discuss their use after solving.

1.OA.3 Apply properties of operations as strategies to add and subtract.

There are a variety of methods to change to an easier problem. These draw on addition of three whole numbers.[1.OA.2] A known addition or subtraction can be used to solve a related addition or subtraction by decomposing one addend and composing it with the other addend. For example, a student can change $8+6$ to the easier $10+4$ by decomposing $6=2+4$ and composing the 2 with the 8 to make 10: $8+6=8+2+4=10+4=14$.

1.OA.2 Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

This method can also be used to subtract by finding an unknown addend: $14-8=\square$, so $8+\square=14$, so $14=8+2+4=8+6$, that is $14-8=6$. Students can think as for adding above (stopping when they reach 14), or they can think of taking 8 from 10, leaving 2 with the 4, which makes 6. One can also decompose with respect to ten: $13-4=13-3-1=10-1=9$, but this can be more difficult than the forward methods.

These make-a-ten methods• have three prerequisites reaching

• Computing 8 + 6 by making a ten

a. 8's partner to 10 is 2, so decompose 6 as 2 and its partner.

b. 2's partner to 6 is 4.

c. 10 + 4 is 14.

back to Kindergarten:

a. knowing the partner that makes 10 for any number (K.OA.4 sets the stage for this),

b. knowing all decompositions for any number below 10 (K.OA.3 sets the stage for this), and

c. knowing all teen numbers as $10 + n$ (e.g., $12 = 10 + 2$, $15 = 10 + 5$, see K.NBT.1 and 1.NBT.2b).

The make-a-ten methods are more difficult in English than in East Asian languages in which teen numbers are spoken as *ten, ten one, ten two, ten three*, etc. In particular, prerequisite c is harder in English because of the irregularities and reversals in the teen number words.•

Another Level 3 method that works for certain numbers is a doubles ± 1 or ± 2 method: $6+7 = 6+(6+1) = (6+6)+1 = 12+1 = 13$. These methods do not connect with place value the way make-a-ten methods do.

The Add To and Take From Start Unknown situations are particularly challenging with the larger numbers students encounter in Grade 1. The situation equation $\square + 6 = 15$ or $\square - 6 = 9$ can be rewritten to provide a solution. Students might use the commutative property of addition to change $\square + 6 = 15$ to $6 + \square = 15$, then count on or use Level 3 methods to compose 4 (to make ten) plus 5 (ones in the 15) to find 9. Students might reverse the action in the situation represented by $\square - 6 = 9$ so that it becomes $9 + 6 = \square$. Or they might use their knowledge that the total is the first number in a subtraction equation and the last number in an addition equation to rewrite the situation equation as a solution equation: $\square - 6 = 9$ becomes $9 + 6 = \square$ or $6 + 9 = \square$.

The difficulty levels in Compare problems differ from those in Put Together/Take Apart and Add To and Take From problems. Difficulties arise from the language issues mentioned before and especially from the opposite language variants where the comparing sentence suggests an operation opposite to that needed for the solution.•

As students progress to Level 2 and Level 3 methods, they no longer need representations that show each quantity as a group of objects. Students now move on to diagrams that use numbers and show relationships between these numbers. These can be extensions of drawings made earlier that did show each quantity as a group of objects. Add To/Take From situations at this point can continue to be represented by equations. Put Together/Take Apart situations can be represented by the example drawings shown in the margin. Compare situations can be represented with tape diagrams showing the compared quantities (one smaller and one larger) and the difference. Other diagrams showing two numbers and the unknown can also be used. Such diagrams are a major step forward because the same diagrams can represent the adding and subtracting situations

K.OA.4 For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.

K.OA.3 Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., $5 = 2+3$ and $5 = 4+1$).

K.NBT.1 Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

1.NBT.2b Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:

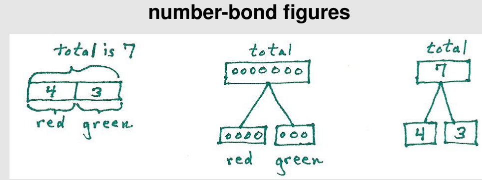
b The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.

• For example, "four" is spoken first in "fourteen," but this order is reversed in the numeral 14.

• Bigger Unknown: *"Fewer" version suggests wrong operation.* Lucy has *B* fewer apples than Julie. Lucy has *A* apples. How many apples does Julie have?

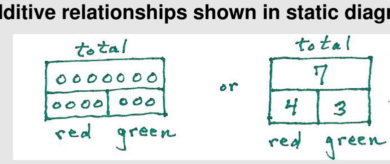
Smaller Unknown. *"More" version suggests wrong operation.* Julie has *B* more apples than Lucy. Julie has *C* apples. How many apples does Lucy have?

Additive relationship shown in tape, part-whole, and number-bond figures



The tape diagram shows the addends as the tapes and the total (indicated by a bracket) as a composition of those tapes. The part-whole diagram and number-bond diagram capture the composing-decomposing action to allow the representation of the total at the top and the addends at the bottom either as drawn quantities or as numbers.

Additive relationships shown in static diagrams



Students sometimes have trouble with static part-whole diagrams because these display a double representation of the total and the addends (the total 7 above and the addends 4 and 3 below), but at a given time in the addition or subtraction situation not all three quantities are present. The action of moving from the total to the addends (or from the addends to the total) in the number-bond diagram reduces this conceptual difficulty.

for all of the kinds of numbers students encounter in later grades (multi-digit whole numbers, fractions, decimals, variables). Students can also continue to represent any situation with a situation equation and connect such equations to diagrams.[MP1] Such connections can help students to solve the more difficult problem situation subtypes by understanding where the totals and addends are in the equation and rewriting the equation as needed.

[MP1] By relating equations and diagrams, students work toward this aspect of MP1: Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs.

Grade 2

Grade 2 students build upon their work in Grade 1 in two major ways.[2.OA.1] They represent and solve situational problems of all three types which involve addition and subtraction within 100 rather than within 20, and they represent and solve two-step situational problems of all three types.

Diagrams used in Grade 1 to show how quantities in the situation are related continue to be useful in Grade 2, and students continue to relate the diagrams to situation equations. Such relating helps students rewrite a situation equation like $\square - 38 = 49$ as $49 + 38 = \square$ because they see that the first number in the subtraction equation is the total. Each addition and subtraction equation has seven related equations. Students can write all of these equations, continuing to connect addition and subtraction, and their experience with equations of various forms.

Because there are so many problem situation subtypes, there are many possible ways to combine such subtypes to devise two-step problems. Because some Grade 2 students are still developing proficiency with the most difficult subtypes, two-step problems should not involve these subtypes. Most work with two-step problems should involve single-digit addends.

Most two-step problems made from two easy subtypes are easy to represent with an equation, as shown in the first two examples to the right. But problems involving a comparison or two middle difficulty subtypes may be difficult to represent with a single equation and may be better represented by successive drawings or some combination of a diagram for one step and an equation for the other (see the last three examples). Students can make up any kinds of two-step problems and share them for solving.

The deep extended experiences students have with addition and subtraction in Kindergarten and Grade 1 culminate in Grade 2 with students becoming fluent in single-digit additions and the related subtractions using the mental Level 2 and 3 strategies as needed.[2.OA.2] So fluency in adding and subtracting single-digit numbers has progressed from numbers within 5 in Kindergarten to within 10 in Grade 1 to within 20 in Grade 2. The methods have also become more advanced.

The word *fluent* is used in the Standards to mean "fast and accurate." Fluency in each grade involves a mixture of just knowing some answers, knowing some answers from patterns (e.g., "adding 0 yields the same number"), and knowing some answers from the use of strategies. It is important to push sensitively and encouragingly toward fluency of the designated numbers at each grade level, recognizing that fluency will be a mixture of these kinds of thinking which may differ across students. The extensive work relating addition and subtraction means that subtraction can frequently be solved by thinking of the related addition, especially for smaller numbers. It is also important that these patterns, strategies and decomposi-

[2.OA.1] Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

Related addition and subtraction equations

$87 - 38 = 49$	$87 - 49 = 38$	$38 + 49 = 87$	$49 + 38 = 87$
$49 = 87 - 38$	$38 = 87 - 49$	$87 = 38 + 49$	$87 = 49 + 38$

Examples of two-step Grade 2 word problems

Two easy subtypes with the same operation, resulting in problems represented as, for example, $9 + 5 + 7 = \square$ or $16 - 8 - 5 = \square$ and perhaps by drawings showing these steps:

> Example for $9 + 5 + 7$: There were 9 blue balls and 5 red balls in the bag. Aki put in 7 more balls. How many balls are in the bag altogether?

Two easy subtypes with opposite operations, resulting in problems represented as, for example, $9 - 5 + 7 = \square$ or $16 + 8 - 5 = \square$ and perhaps by drawings showing these steps:

> Example for $9 - 5 + 7$: There were 9 carrots on the plate. The girls ate 5 carrots. Mother put 7 more carrots on the plate. How many carrots are there now?

One easy and one middle difficulty subtype:

> For example: Maria has 9 apples. Corey has 4 fewer apples than Maria. How many apples do they have in all?

> For example: The zoo had 7 cows and some horses in the big pen. There were 15 animals in the big pen. Then 4 more horses ran into the big pen. How many horses are there now?

Two middle difficulty subtypes:

> For example: There were 9 boys and some girls in the park. In all, 15 children were in the park. Then some more girls came. Now there are 14 girls in the park. How many more girls came to the park?

[2.OA.2] Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers.

tions still be available in Grade 3 for use in multiplying and dividing and in distinguishing adding and subtracting from multiplying and dividing. So the important press toward fluency should also allow students to fall back on earlier strategies when needed. By the end of the K–2 grade span, students have sufficient experience with addition and subtraction to know single-digit sums from memory;[2.OA.2] as should be clear from the foregoing, this is not a matter of instilling facts divorced from their meanings, but rather as an outcome of a multi-year process that heavily involves the interplay of practice and reasoning.

Extensions to other standard domains and to higher grades In Grades 2 and 3, students continue and extend their work with adding and subtracting situations to length situations[2.MD.5,2.MD.6] (addition and subtraction of lengths is part of the transition from whole number addition and subtraction to fraction addition and subtraction) and to bar graphs.[2.MD.10,3.MD.3] Students solve two-step[3.OA.8] and multistep[4.OA.3] problems involving all four operations. In Grades 3, 4, and 5, students extend their understandings of addition and subtraction problem types in Table 1 to situations that involve fractions and decimals. Importantly, the situational meanings for addition and subtraction remain the same for fractions and decimals as for whole numbers.

[2.OA.2] Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers.

[2.MD.5] Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

[2.MD.6] Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

[2.MD.10] Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph.

[3.MD.3] Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs.

[3.OA.8] Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

[4.OA.3] Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

Summary of K–2 Operations and Algebraic Thinking

Kindergarten Students in Kindergarten work with three kinds of problem situations: Add To with Result Unknown; Take From with Result Unknown; and Put Together/Take Apart with Total Unknown and Both Addends Unknown. The numbers in these problems involve addition and subtraction within 10. Students represent these problems with concrete objects and drawings, and they find the answers by counting (Level 1 method). More specifically,

- For Add To with Result Unknown, they make or draw the starting set of objects and the change set of objects, and then they count the total set of objects to give the answer.
- For Take From with Result Unknown, they make or draw the starting set and "take away" the change set; then they count the remaining objects to give the answer.
- For Put Together/Take Apart with Total Unknown, they make or draw the two addend sets, and then they count the total number of objects to give the answer.

Grade 1 Students in Grade 1 work with all of the problem situations, including all subtypes and language variants. The numbers in these problems involve additions involving single-digit addends, and the related subtractions. Students represent these problems with math drawings and with equations.

Students master the majority of the problem types. They might sometimes use trial and error to find the answer, or they might just know the answer based on previous experience with the given numbers. But as a general method they learn how to find answers to these problems by counting on (a Level 2 method), and they understand and use this method.[1.OA.5,1.OA.6] Students also work with Level 3 methods that change a problem to an easier equivalent problem.[1.OA.3,1.OA.6] The most important of these Level 3 methods involve making a ten, because these methods connect with the place value concepts students are learning in this grade (see the NBT Progression) and work for any numbers. Students also solve the easier problem subtypes with these Level 3 methods.

The four problem subtypes that Grade 1 students should work with, but need not master, are:

- Add To with Start Unknown
- Take From with Start Unknown
- Compare with Bigger Unknown using "fewer" language (misleading language suggesting the wrong operation)
- Compare with Smaller Unknown using "more" language (misleading language suggesting the wrong operation)

1.OA.5 Relate counting to addition and subtraction (e.g., by counting on 2 to add 2).

1.OA.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8+6=8+2+4=10+4=14$); decomposing a number leading to a ten (e.g., $13-4=13-3-1=10-1=9$); using the relationship between addition and subtraction (e.g., knowing that $8+4=12$, one knows $12-8=4$); and creating equivalent but easier or known sums (e.g., adding $6+7$ by creating the known equivalent $6+6+1=12+1=13$).

1.OA.3 Apply properties of operations as strategies to add and subtract.

1.OA.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8+6=8+2+4=10+4=14$); decomposing a number leading to a ten (e.g., $13-4=13-3-1=10-1=9$); using the relationship between addition and subtraction (e.g., knowing that $8+4=12$, one knows $12-8=4$); and creating equivalent but easier or known sums (e.g., adding $6+7$ by creating the known equivalent $6+6+1=12+1=13$).

Grade 2 Students in Grade 2 master all of the problem situations and all of their subtypes and language variants. The numbers in these problems involve addition and subtraction within 100. They represent these problems with diagrams and/or equations. For problems involving addition and subtraction within 20, more students master Level 3 methods; increasingly for addition problems, students might just know the answer (by end of Grade 2, students know all sums of two-digit numbers from memory[2.OA.2]). For other problems involving numbers to 100, Grade 2 students use their developing place value skills and understandings to find the answer (see the NBT Progression). Students work with two-step problems, especially with single-digit addends, but do not work with two-step problems in which both steps involve the most difficult problem subtypes and variants.

[2.OA.2] Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers.

Grade 3

Students focus on understanding the meaning and properties of multiplication and division and on finding products of single-digit multiplying and related quotients.[3.OA.1–7] These skills and understandings are crucial; students will rely on them for years to come as they learn to multiply and divide with multi-digit whole number and to add, subtract, multiply and divide with fractions and with decimals. Note that mastering this material, and reaching fluency in single-digit multiplications and related divisions with understanding,[3.OA.7] may be quite time consuming because there are no general strategies for multiplying or dividing all single-digit numbers as there are for addition and subtraction. Instead, there are many patterns and strategies dependent upon specific numbers. So it is imperative that extra time and support be provided if needed.

Common types of multiplication and division situations. Common multiplication and division situations are shown in Table 3. There are three major types, shown as rows of Table 3. The Grade 3 standards focus on Equal Groups and on Arrays.• As with addition and subtraction, each multiplication or division situation involves three quantities, each of which can be the unknown. Because there are two factors and one product in each situation (product = factor × factor), each type has one subtype solved by multiplication (Unknown Product) and two unknown factor subtypes solved by division.

3.OA.1 Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in 5 groups of 7 objects each.

3.OA.2 Interpret whole-number quotients of whole numbers, e.g., interpret $56 \div 8$ as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each.

3.OA.3 Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

3.OA.4 Determine the unknown whole number in a multiplication or division equation relating three whole numbers.

3.OA.5 Apply properties of operations as strategies to multiply and divide.

3.OA.6 Understand division as an unknown-factor problem.

3.OA.7 Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

• Multiplicative Compare situations are more complex than Equal Groups and Arrays, and must be carefully distinguished from additive Compare problems. Multiplicative comparison first enters the Standards at Grade 4.[4.OA.1] For more information on multiplicative Compare problems, see the Grade 4 section of this progression.

4.OA.1 Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

Table 3: Multiplication and division situations

	$A \times B = \square$	$A \times \square = C$ and $C \div A = \square$	$\square \times B = C$ and $C \div B = \square$
Equal Groups of Objects	Unknown Product There are A bags with B plums in each bag. How many plums are there in all?	Group Size Unknown If C plums are shared equally into A bags, then how many plums will be in each bag?	Number of Groups Unknown If C plums are to be packed B to a bag, then how many bags are needed?
Arrays of Objects		*Equal groups language*	
	Unknown Product There are A rows of apples with B apples in each row. How many apples are there?	Unknown Factor If C apples are arranged into A equal rows, how many apples will be in each row?	Unknown Factor If C apples are arranged into equal rows of B apples, how many rows will there be?
		Row and column language	
	Unknown Product The apples in the grocery window are in A rows and B columns. How many apples are there?	Unknown Factor If C apples are arranged into an array with A rows, how many columns of apples are there?	Unknown Factor If C apples are arranged into an array with B columns, how many rows are there?
Compare		$A > 1$	
	Larger Unknown A blue hat costs $\$B$. A red hat costs A times as much as the blue hat. How much does the red hat cost?	Smaller Unknown A red hat costs $\$C$ and that is A times as much as a blue hat costs. How much does a blue hat cost?	Multiplier Unknown A red hat costs $\$C$ and a blue hat costs $\$B$. How many times as much does the red hat cost as the blue hat?
		$A < 1$	
	Smaller Unknown A blue hat costs $\$B$. A red hat costs A as much as the blue hat. How much does the red hat cost?	Larger Unknown A red hat costs $\$C$ and that is A of the cost of a blue hat. How much does a blue hat cost?	Multiplier Unknown A red hat costs $\$C$ and a blue hat costs $\$B$. What fraction of the cost of the blue hat is the cost of the red hat?

Adapted from box 2–4 of *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, National Research Council, 2009, pp. 32–33.

Notes

Equal groups problems can also be stated in terms of columns, exchanging the order of A and B, so that the same array is described. For example: There are B columns of apples with A apples in each column. How many apples are there?

In the row and column situations (as with their area analogues), number of groups and group size are not distinguished.

Multiplicative Compare problems appear first in Grade 4, with whole-number values for A, B, and C, and with the "times as much" language in the table. In Grade 5, unit fractions language such as "one third as much" may be used. Multiplying and unit fraction language change the subject of the comparing sentence, e.g., "A red hat costs A times as much as the blue hat" results in the same comparison as "A blue hat costs $1/A$ times as much as the red hat," but has a different subject.

In Equal Groups, the roles of the factors differ. One factor is the number of objects in a group (like any quantity in addition and subtraction situations), and the other is a multiplier that indicates the number of groups. So, for example, 4 groups of 3 objects is arranged differently than 3 groups of 4 objects. Thus there are two kinds of division situations depending on which factor is the unknown (the number of objects in each group or the number of groups). In the Array situations, the roles of the factors do not differ. One factor tells the number of rows in the array, and the other factor tells the number of columns in the situation. But rows and columns depend on the orientation of the array. If an array is rotated 90°, the rows become columns and the columns become rows. This is useful for seeing the commutative property for multiplication[3.OA.5] in rectangular arrays and areas. This property can be seen to extend to Equal Group situations when Equal Group situations are related to arrays by arranging each group in a row and putting the groups under each other to form an array. Array situations can be seen as Equal Group situations if each row or column is considered as a group. Relating Equal Group situations to Arrays, and indicating rows or columns within arrays, can help students see that a corner object in an array (or a corner square in an area model) is not double counted: at a given time, it is counted as part of a row or as a part of a column but not both.

As noted in Table 3, row and column language can be difficult. The Array problems given in the table are of the simplest form in which a row is a group and Equal Groups language is used ("with 6 apples in each row"). Such problems are a good transition between the Equal Groups and array situations and can support the generalization of the commutative property discussed above. Problems in terms of "rows" and "columns," e.g., "The apples in the grocery window are in 3 rows and 6 columns," are difficult because of the distinction between the number of things *in a* row and the number *of* rows. There are 3 rows but the number of columns (6) tells how many are in each row. There are 6 columns but the number of rows (3) tells how many are in each column. Students do need to be able to use and understand these words, but this understanding can grow over time while students also learn and use the language in the other multiplication and division situations.

Variations of each type that use measurements instead of discrete objects are given in the Measurement and Data Progression. Grade 2 standards focus on length measurement[2.MD.1–4] and Grade 3 standards focus on area measurement.[3.MD.5–7] The measurement examples are more difficult than are the examples about discrete objects, so these should follow problems about discrete objects. Area problems where regions are partitioned by unit squares are foundational for Grade 3 standards because area is used as a model for single-digit multiplication and division strategies,[3.MD.7] in Grade 4 as a model for multi-digit multiplication and division and in Grade 5 and Grade 6 as a model for multiplication and division of decimals

3.OA.5 Apply properties of operations as strategies to multiply and divide.

2.MD.1 Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.

2.MD.2 Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.

2.MD.3 Estimate lengths using units of inches, feet, centimeters, and meters.

2.MD.4 Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

3.MD.5 Recognize area as an attribute of plane figures and understand concepts of area measurement.

3.MD.6 Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

3.MD.7 Relate area to the operations of multiplication and addition.

and of fractions.[5.NBT.6] The distributive property is central to all of these uses and will be discussed later.

The top row of Table 3 shows the usual order of writing multiplications of Equal Groups in the United States. The equation $3 \times 6 = \square$ means how many are in 3 groups of 6 things each: three sixes. But in many other countries the equation $3 \times 6 = \square$ means how many are 3 things taken 6 times (6 groups of 3 things each): six threes. Some students bring this interpretation of multiplication equations into the classroom. So it is useful to discuss the different interpretations and allow students to use whichever is used in their home. This is a kind of linguistic commutativity that precedes the reasoning discussed above arising from rotating an array. These two sources of commutativity can be related when the rotation discussion occurs.

5.NBT.6 Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Levels in problem representation and solution Multiplication and division problem representations and solution methods can be considered as falling within three levels related to the levels for addition and subtraction (see Appendix). Level 1 is making and counting all of the quantities involved in a multiplication or division. As before, the quantities can be represented by objects or with a diagram, but a diagram affords reflection and sharing when it is drawn on the board and explained by a student. The Grade 2 standards 2.OA.3 and 2.OA.4 are at this level but set the stage for Level 2. Standard 2.OA.3 relates doubles additions up to 20 to the concept of odd and even numbers and to counting by 2s (the easiest count-by in Level 2) by pairing and counting by 2s the things in each addend. 2.OA.4 focuses on using addition to find the total number of objects arranged in rectangular arrays (up to 5 by 5).

2.OA.3 Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.

2.OA.4 Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.

Level 2 is repeated counting on by a given number, such as for 3: $3, 6, 9, 12, 15, 18, 21, 24, 27, 30$. The count-bys give the running total. The number of 3s said is tracked with fingers or a visual or physical (e.g., head bobs) pattern. For 8×3, you know the number of 3s and count by 3 until you reach 8 of them. For $24 \div 3$, you count by 3 until you hear 24, then look at your tracking method to see how many 3s you have. Because listening for 24 is easier than monitoring the tracking method for 8 3s to stop at 8, dividing can be easier than multiplying.

Supporting Level 2 methods with arrays

Small arrays (up to 5×5) support seeing and beginning to learn the Level 2 count-bys for the first five equal groups of the small numbers 2 through 5 if the running total is written to the right of each row (e.g., $3, 6, 9, 12, 15$). Students may write repeated additions and then count by ones without the objects, often emphasizing each last number said for each group. Grade 3 students can be encouraged to move as early as possible from equal grouping or array models that show all of the quantities to similar representations using diagrams that show relationships of numbers because diagrams are faster and less error-prone and support methods at Level 2 and Level 3. Some demonstrations of methods or of properties may need to fall back to initially showing all quantities along with a diagram.

The difficulty of saying and remembering the count-by for a given number depends on how closely related it is to 10, the base for our written and spoken numbers. For example, the count-by sequence for 5 is easy, but the count-by sequence for 7 is difficult. Decomposing with respect to a ten can be useful in going over a decade within a count-by. For example in the count-by for 7, students might use the following mental decompositions of 7 to compose up to and then go over the next decade, e.g., $14 + 7 = 14 + 6 + 1 = 20 + 1 = 21$. The count-by sequence can also be said with the factors, such as "one times three is *three*, two times three is *six*, three times three is *nine*, etc." Seeing as well as hearing the count-bys and the equations for

Composing up to, then over the next decade

$$7 \quad 14 \quad 21 \quad 28 \quad 35 \quad 42 \quad 49 \quad 56 \quad 63 \quad 70$$

$$6+1 \qquad 2+5 \quad 5+2 \qquad 1+6 \quad 4+3$$

There is an initial $3 + 4$ for $7 + 7$ that completes the reversing pattern of the partners of 7 involved in these mental decompositions with respect to the decades.

the multiplications or divisions can be helpful.

Level 3 methods use the associative property or the distributive property to compose and decompose. These compositions and decompositions may be additive (as for addition and subtraction) or multiplicative. For example, students multiplicatively compose or decompose:

> 4×6 is easier to count by 3 eight times:
>
> $$4 \times 6 = 4 \times (2 \times 3) = (4 \times 2) \times 3 = 8 \times 3.$$

Students may know a product 1 or 2 ahead of or behind a given product and say:

> I know 6×5 is 30, so 7×5 is $30 + 5$ more which is 35.
> This implicitly uses the distributive property:
>
> $$7 \times 5 = (6 + 1) \times 5 = 6 \times 5 + 1 \times 3 = 30 + 5 = 35.$$

Students may decompose a product that they do not know in terms of two products they know (for example, 4×7 shown in the margin).

Students may not use the properties explicitly (for example, they might omit the second two steps), but classroom discussion can identify and record properties in student reasoning. An area diagram can support such reasoning.

The $5 + n$ pattern students used earlier for additions can now be extended to show how 6, 7, 8, and 9 times a number are $5 + 1$, $5 + 2$, $5 + 3$, and $5 + 4$ times that number. These patterns are particularly easy to do mentally for the numbers 4, 6, and 8. The 9s have particularly rich patterns based on $9 = 10 - 1$. The pattern of the tens digit in the product being 1 less than the multiplier, the ones digit in the product being 10 minus the multiplier, and that the digits in nines products sum to 9 all come from this pattern.

There are many opportunities to describe and reason about the many patterns involved in the Level 2 count-bys and in the Level 3 composing and decomposing methods. There are also patterns in multiplying by 0 and by 1. These need to be differentiated from the patterns for adding 0 and adding 1 because students often confuse these three patterns: $n + 0 = n$ but $n \times 0 = 0$, and $n \times 1$ is the pattern that does not change n (because $n \times 1 = n$). Patterns make multiplication by some numbers easier to learn than multiplication by others, so approaches may teach multiplications and divisions in various orders depending on what numbers are seen as or are supported to be easiest.

Multiplications and divisions can be learned at the same time and can reinforce each other. Level 2 methods can be particularly easy for division, as discussed above. Level 3 methods may be more difficult for division than for multiplication.

Throughout multiplication and division learning, students gain fluency and begin to know certain products and unknown factors.

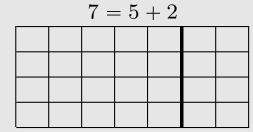
Decomposing 4×7

$$\begin{aligned} 4 \times 7 &= 4 \times (5 + 2) \\ &= (4 \times 5) + (4 \times 2) \\ &= 20 + 8 \\ &= 28 \end{aligned}$$

Supporting reasoning with area diagram

$7 = 5 + 2$

4



$20 + 8 = 28$

$4 \times 5 + 4 \times 2 = 4 \times 7$

The $5 + n$ pattern for multiplying the numbers 4, 6, and 8

n		$4 \times n$		$6 \times n$		$8 \times n$	
1	$5+1$	4	24	6	36	8	48
2	$5+2$	8	28	12	42	16	56
3	$5+3$	12	32	18	48	24	64
4	$5+4$	16	36	24	54	32	72
5	$5+5$	20	40	30	60	40	80

Patterns in multiples of 9

$1 \times 9 = 9$

$2 \times 9 = 2 \times (10 - 1) = (2 \times 10) - (2 \times 1) = 20 - 2 = 18$

$3 \times 9 = 3 \times (10 - 1) = (3 \times 10) - (3 \times 1) = 30 - 3 = 27,$ etc

All of the understandings of multiplication and division situations, of the levels of representation and solving, and of patterns need to culminate by the end of Grade 3 in fluent multiplying and dividing of all single-digit numbers and 10.[3.OA.7] Such fluency may be reached by becoming fluent for each number (e.g., the 2s, the 5s, etc.) and then extending the fluency to several, then all numbers mixed together. Organizing practice so that it focuses most heavily on understood but not yet fluent products and unknown factors can speed learning. To achieve this by the end of Grade 3, students must begin working toward fluency for the easy numbers as early as possible. Because an unknown factor (a division) can be found from the related multiplication, the emphasis at the end of the year is on knowing from memory all products of two one-digit numbers. As should be clear from the foregoing, this isn't a matter of instilling facts divorced from their meanings, but rather the outcome of a carefully designed learning process that heavily involves the interplay of practice and reasoning. All of the work on how different numbers fit with the base-ten numbers culminates in these "just know" products and is necessary for learning products. Fluent dividing for all single-digit numbers, which will combine just knows, knowing from a multiplication, patterns, and best strategy, is also part of this vital standard.

[3.OA.7] Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

Using a letter for the unknown quantity, the order of operations, and two-step word problems with all four operations Students in Grade 3 begin the step to formal algebraic language by using a letter for the unknown quantity in expressions or equations for one- and two-step problems.[3.OA.8] But the symbols of arithmetic, $\times$ or $\cdot$ or $*$ for multiplication and $\div$ or / for division, continue to be used in Grades 3, 4, and 5.

[3.OA.8] Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

Understanding and using the associative and distributive properties (as discussed above) requires students to know two conventions for reading an expression that has more than one operation:

1. Do the operation inside the parentheses before an operation outside the parentheses (the parentheses can be thought of as hands curved around the symbols and grouping them).

2. If a multiplication or division is written next to an addition or subtraction, imagine parentheses around the multiplication or division (it is done before these operations). At Grades 3 through 5, parentheses can usually be used for such cases so that fluency with this rule can wait until Grade 6.

MP7 Making use of structure to make computation easier:

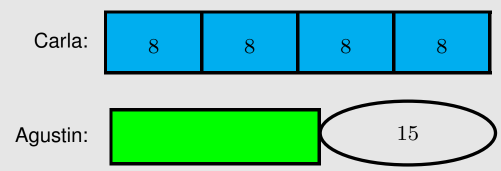
$$13 + 29 + 77 + 11 = (13 + 77) + (29 + 11)$$

These conventions are often called the Order of Operations and can seem to be a central aspect of algebra. But actually they are just simple "rules of the road" that allow expressions involving more than one operation to be interpreted unambiguously and thus are connected with the mathematical practice of communicating precisely.[MP6] Use of parentheses is important in displaying structure and thus is connected with the mathematical practice of making use of structure.[MP7] Parentheses are important in expressing the associative and especially the distributive properties. These properties are at the heart of Grades 3 to 5 because they are used in the Level 3 multiplication and division strategies, in multi-digit and decimal multiplication and division, and in all operations with fractions.

As with two-step problems at Grade 2,[2.OA.1, 2.MD.5] which involve only addition and subtraction, the Grade 3 two-step word problems vary greatly in difficulty and ease of representation. More difficult problems may require two steps of representation and solution rather than one. Use of two-step problems involving easy or middle difficulty adding and subtracting within 1,000 or one such adding or subtracting with one step of multiplication or division can help to maintain fluency with addition and subtraction while giving the needed time to the major Grade 3 multiplication and division standards.

A two-step problem with diagram showing problem situation and equations showing the two parts

Carla has 4 packages of silly bands. Each package has 8 silly bands in it. Agustin is supposed to get 15 fewer silly bands than Carla. How many silly bands should Agustin get?



C = number of Carla's silly bands
A = number of Agustin's silly bands

$$C = 4 \times 8 = 32$$

$$A + 15 = C$$
$$A + 15 = 32$$
$$A = 17$$

Students may be able to solve this problem without writing such equations.

Grade 4

Multiplication Compare Consider two diving boards, one 40 feet high, the other 8 feet high. Students in earlier grades learned to compare these heights in an additive sense—"This one is 32 feet higher than that one"—by solving additive Compare problems[2.OA.1] and using addition and subtraction to solve word problems involving length.[2.MD.5] Students in Grade 4 learn to compare these quantities multiplicatively as well: "This one is 5 times as high as that one."[4.OA.1,4.OA.2,4.MD.1,4.MD.2] In an additive comparison, the underlying question is *what amount would be added to one quantity* in order to result in the other. In a multiplicative comparison, the underlying question is *what factor would multiply one quantity* in order to result in the other. Multiplication Compare situations are shown in Table 3.

Language can be difficult in Multiplication Compare problems. The language used in the three examples in Table 3 is fairly simple, e.g., "A red hat costs 3 times as much as the blue hat." Saying the comparing sentence in the opposite way is more difficult. It could be said using division, e.g., "The cost of a red hat divided by 3 is the cost of a blue hat." It could also be said using a unit fraction, e.g., "A blue hat costs one-third as much as a red hat"; note however that multiplying by a fraction in not an expectation of the Standards in Grade 4. In any case, many languages do not use either of these options for saying the opposite comparison. They use the terms *three times more than* and *three times less than* to describe opposite multiplicative comparisons. These did not used to be acceptable usages in English because they mix the multiplicative and additive comparisons and are ambiguous. If the cost of a red hat is three times more than a blue hat that costs $5, does a red hat cost $15 (three times as much) or $20 (three times more than: a difference that is three times as much)? However, the terms *three times more than* and *three times less than* are now appearing frequently in newspapers and other written materials. It is recommended to discuss these complexities with Grade 4 students while confining problems that appear on tests or in multi-step problems to the well-defined multiplication language in Table 3. The tape diagram for the additive Compare situation that shows a smaller and a larger tape can be extended to the multiplication Compare situation.

Fourth graders extend problem solving to multi-step word problems using the four operations posed with whole numbers. The same limitations discussed for two-step problems concerning representing such problems using equations apply here. Some problems might easily be represented with a single equation, and others will be more sensibly represented by more than one equation or a diagram and one or more equations. Numbers can be those in the Grade 4 standards, but the number of steps should be no more than three and involve only easy and medium difficulty addition and subtraction problems.

[2.OA.1] Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

[2.MD.5] Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

[4.OA.1] Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

[4.OA.2] Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison.

[4.MD.1] Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.

[4.MD.2] Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

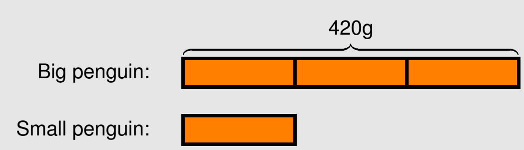
Tape diagram used to solve the Compare problem in Table 3

B is the cost of a blue hat in dollars
R is the cost of a red hat in dollars

| $6 | | | $3 \times B = R$ |
| $6 | $6 | $6 | $3 \times \$6 = \$18$ |

A tape diagram used to solve a Compare problem

A big penguin will eat 3 times as much fish as a small penguin. The big penguin will eat 420 grams of fish. All together, how much will the two penguins eat?



B = number of grams the big penguin eats
S = number of grams the small penguin eats

$$3 \cdot S = B$$
$$3 \cdot S = 420$$
$$S = 140$$

$$S + B = 140 + 420 = 560$$

Remainders In problem situations, students must interpret and use remainders with respect to context.[4.OA.3] For example, what is the smallest number of busses that can carry 250 students, if each bus holds 36 students? The whole number quotient in this case is 6 and the remainder is 34; the equation $250 = 6 \times 36 + 34$ expresses this result and corresponds to a picture in which 6 busses are completely filled while a seventh bus carries 34 students. Notice that the answer to the stated question (7) differs from the whole number quotient.

On the other hand, suppose 250 pencils were distributed among 36 students, with each student receiving the same number of pencils. What is the largest number of pencils each student could have received? In this case, the answer to the stated question (6) is the same as the whole number quotient. If the problem had said that the teacher got the remaining pencils and asked how many pencils the teacher got, then the remainder would have been the answer to the problem.

[4.OA.3] Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

Factors, multiples, and prime and composite numbers Students extend the idea of decomposition to multiplication and learn to use the term *multiple*.[4.OA.4] Any whole number is a multiple of each of its factors, so for example, 21 is a multiple of 3 and a multiple of 7 because $21 = 3 \cdot 7$. A number can be multiplicatively decomposed into equal groups and expressed as a product of these two factors (called factor pairs). A prime number has only one and itself as factors. A composite number has two or more factor pairs. Students examine various patterns in factor pairs by finding factor pairs for all numbers 1 to 100 (e.g., no even number other than 2 will be prime because it always will have a factor pair including 2). To find all factor pairs for a given number, students can search systematically, by checking if 2 is a factor, then 3, then 4, and so on, until they start to see a "reversal" in the pairs (for example, after finding the pair 6 and 9 for 54, students will next find the reverse pair, 9 and 6; all subsequent pairs will be reverses of previously found pairs). Students understand and use of the concepts and language in this area, but need not be fluent in finding all factor pairs. Determining whether a given whole number in the range 1 to 100 is a multiple of a given one-digit number is a matter of interpreting prior knowledge of division in terms of the language of multiples and factors.

[4.OA.4] Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.

Generating and analyzing patterns This standard[4.OA.5] begins a small focus on reasoning about number or shape patterns, connecting a rule for a given pattern with its sequence of numbers or shapes. Patterns that consist of repeated sequences of shapes or growing sequences of designs can be appropriate for the grade. For example, students could examine a sequence of dot designs in which each design has 4 more dots than the previous one and they could reason about how the dots are organized in the design to determine the

[4.OA.5] Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself.

total number of dots in the 100th design. In examining numerical sequences, fourth graders can explore rules of repeatedly adding the same whole number or repeatedly multiplying by the same whole number. Properties of repeating patterns of shapes can be explored with division. For example, to determine the 100th shape in a pattern that consists of repetitions of the sequence "square, circle, triangle," the fact that when we divide 100 by 3 the whole number quotient is 33 with remainder 1 tells us that after 33 full repeats, the 99th shape will be a triangle (the last shape in the repeating pattern), so the 100th shape is the first shape in the pattern, which is a square. Notice that the Standards do not require students to infer or guess the underlying rule for a pattern, but rather ask them to generate a pattern from a given rule and identify features of the given pattern.

Grade 5

As preparation for the Expressions and Equations Progression in the middle grades, students in Grade 5 begin working more formally with expressions.[5.OA.1, 5.OA.2] They write expressions to express a calculation, e.g., writing $2 \times (8+7)$ to express the calculation "add 8 and 7, then multiply by 2." They also evaluate and interpret expressions, e.g., using their conceptual understanding of multiplication to interpret $3 \times (18932 + 921)$ as being three times as large as $18932 + 921$, without having to calculate the indicated sum or product. Thus, students in Grade 5 begin to think about numerical expressions in ways that prefigure their later work with variable expressions (e.g., three times an unknown length is $3 \cdot L$). In Grade 5, this work should be viewed as exploratory rather than for attaining mastery; for example, expressions should not contain nested grouping symbols, and they should be no more complex than the expressions one finds in an application of the associative or distributive property, e.g., $(8+27)+2$ or $(6 \times 30) + (6 \times 7)$. Note however that the numbers in expressions need not always be whole numbers.

5.OA.1 Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.

5.OA.2 Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them.

Students extend their Grade 4 pattern work by working briefly with two numerical patterns that can be related and examining these relationships within sequences of ordered pairs and in the graphs in the first quadrant of the coordinate plane.[5.OA.3] This work prepares students for studying proportional relationships and functions in middle school.

5.OA.3 Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane.

Connections to NF and NBT in Grades 3 through 5

Students extend their whole number work with adding and subtracting and multiplying and dividing situations to decimal numbers and fractions. Each of these extensions can begin with problems that include all of the subtypes of the situations in Tables 1 and 2. The operations of addition, subtraction, multiplication, and division continue to be used in the same way in these problem situations when they are extended to fractions and decimals (although making these extensions is not automatic or easy for all students). The connections described for Kindergarten through Grade 3 among word problem situations, representations for these problems, and use of properties in solution methods are equally relevant for these new kinds of numbers. Students use the new kinds of numbers, fractions and decimals, in geometric measurement and data problems and extend to some two-step and multi-step problems involving all four operations. In order to keep the difficulty level from becoming extreme, there should be a tradeoff between the algebraic or situational complexity of any given problem and its computational difficulty taking into account the kinds of numbers involved.

As students' notions of quantity evolve and generalize from discrete to continuous during Grades 3–5, their notions of multiplication evolves and generalizes. This evolution deserves special attention because it begins in OA but ends in NF. Thus, the concept of multiplication begins in Grade 3 with an entirely discrete notion of "equal groups."[3.OA.1] By Grade 4, students can also interpret a multiplication equation as a statement of comparison involving the notion "times as much."[4.OA.1] This notion has more affinity to continuous quantities, e.g., $3 = 4 \times \frac{3}{4}$ might describe how 3 cups of flour are 4 times as much as $\frac{3}{4}$ cup of flour.[4.NF.4,4.MD.2] By Grade 5, when students multiply fractions in general,[5.NF.4] products can be larger or smaller than either factor, and multiplication can be seen as an operation that "stretches or shrinks" by a scale factor.[5.NF.5] This view of multiplication as scaling is the appropriate notion for reasoning multiplicatively with continuous quantities.

[3.OA.1] Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in 5 groups of 7 objects each.

[4.OA.1] Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

[4.NF.4] Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

[4.MD.2] Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

[5.NF.4] Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

[5.NF.5] Interpret multiplication as scaling (resizing), by:

Where the Operations and Algebraic Thinking Progression is heading

Connection to the Number System The properties of and relationships between operations that students worked with in Grades K–5 will become even more prominent in extending arithmetic to systems that include negative numbers; meanwhile the meanings of the operations will continue to evolve, e.g., subtraction will become "adding the opposite."

Connection to Expressions and Equations In Grade 6, students will begin to view expressions not just as calculation recipes but as entities in their own right, which can be described in terms of their parts. For example, students see $8 * (5 + 2)$ as the product of 8 with the sum $5 + 2$. In particular, students must use the conventions for order of operations to *interpret* expressions, not just to evaluate them. Viewing expressions as entities created from component parts is essential for seeing the structure of expressions in later grades and using structure to reason about expressions and functions.

As noted above, the foundation for these later competencies is laid in Grade 5 when students write expressions to record a "calculation recipe" without actually evaluating the expression, use parentheses to formulate expressions, and examine patterns and relationships numerically and visually on a coordinate plane graph.[5.OA.1,5.OA.2] Before Grade 5, student thinking that also builds toward the Grade 6 EE work is focusing on the expressions on each side of an equation, relating each expression to the situation, and discussing the situational and mathematical vocabulary involved to deepen the understandings of expressions and equations.

5.OA.1 Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.

5.OA.2 Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them.

In Grades 6 and 7, students begin to explore the systematic algebraic methods used for solving algebraic equations. Central to these methods are the relationships between addition and subtraction and between multiplication and division, emphasized in several parts of this Progression and prominent also in the 6–8 Progression for the Number System. Students' varied work throughout elementary school with equations with unknowns in all locations and in writing equations to decompose a given number into many pairs of addends or many pairs of factors are also important foundations for understanding equations and for solving equations with algebraic methods. Of course, any method of solving, whether systematic or not, relies on an understanding of what solving itself is—namely, a process of answering a question: which values from a specified set, if any, make the equation true?[6.EE.5]

6.EE.5 Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

Students represent and solve word problems with equations involving one unknown quantity in K through 5. The quantity was expressed by a □ or other symbol in K–2 and by a letter in Grades 3 to 5. Grade 6 students continue the K–5 focus on representing a problem situation using an equation (a situation equation) and then (for the more difficult situations) writing an equivalent equation that is easier to solve (a solution equation). Grade 6 students discuss their reasoning more explicitly by focusing on the structures of expressions and using the properties of operations explicitly. Some of the math drawings that students have used in K through 5 to represent problem situations continue to be used in the middle grades. These can help students throughout the grades deepen the connections they make among the situation and problem representations by a drawing and/or by an equation, and support the informal K–5 and increasingly formal 6–8 solution methods arising from understanding the structure of expressions and equations.

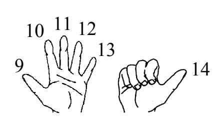
Appendix. Methods used for solving single-digit addition and subtraction problems

Level 1. Direct Modeling by Counting All or Taking Away.

Represent situation or numerical problem with groups of objects, a drawing, or fingers. Model the situation by composing two addend groups or decomposing a total group. Count the resulting total or addend.

Adding ($8 + 6 = \square$): Represent each addend by a group of objects. Put the two groups together. Count the total. Use this strategy for Add To/Result Unknown and Put Together/Total Unknown.

Subtracting ($14 - 8 = \square$): Represent the total by a group of objects. Take the known addend number of objects away. Count the resulting group of objects to find the unknown added. Use this strategy for Take From/Result Unknown.

Levels	**8 + 6 = 14**	**14 – 8 = 6**
Level 1: Count all	Count All a: 1 2 3 4 5 6 7 8 ○○○○○ ○○○ b: 1 2 3 4 5 6 ○○○○○ ○ c: 1 2 3 4 5 6 7 8 9 10 11 12 13 14	Take Away a: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 ~~○○○○○ ○○○~~○○ ○○○○ b: 1 2 3 4 5 6 7 8 c: 1 2 3 4 5 6
Level 2: Count on	Count On 8 (○○○○○ ○○○) → 8 ○○○○○ ○ 9 10 11 12 13 14	To solve 14 – 8 I count on 8 + ? = 14 I took away 8  8 to 14 is 6 so 14 – 8 = 6
Level 3: Recompose Make a ten (general): one addend breaks apart to make 10 with the other addend Make a ten (from 5's within each addend)	Recompose: Make a Ten (○○○○○ ○○○ ○○)○○○ ○ 10 + 4 (○○○○○) (○○○) (○○○○○) (○) 10 + 4	14 – 8: I make a ten for 8 + ? = 14 ~~○○○○○ ○○○~~○○ ○○○○ 8 + 2 + 4 6 8 + 6 = 14
Doubles ± *n*	6 + 8 = 6 + 6 + 2 = 12 + 2 = 14	

Note: Many children attempt to count down for subtraction, but counting down is difficult and error-prone.
Children are much more successful with counting on; it makes subtraction as easy as addition.

Level 2. Counting On.

Embed an addend within the total (the addend is perceived simultaneously as an addend and as part of the total). Count this total but abbreviate the counting by omitting the count of this addend; instead, begin with the number word of this addend. Some method of keeping track (fingers, objects, mentally imaged objects, body motions, other count words) is used to monitor the count.

For addition, the count is stopped when the amount of the remaining addend has been counted. The last number word is the total. For subtraction, the count is stopped when the total occurs in the count. The tracking method indicates the difference (seen as an unknown addend).

Counting on can be used to find the total or to find an addend. These look the same to an observer. The difference is what is monitored: the total or the known addend. Some students count down to solve subtraction problems, but this method is less accurate and more difficult than counting on. Counting on is not a rote method. It requires several connections between cardinal and counting meanings of the number words and extended experience with Level 1 methods in Kindergarten.

> Adding (e. g., $8 + 6 = \square$) uses counting on to find a total: One counts on from the first addend (or the larger number is taken as the first addend). Counting on is monitored so that it stops when the second addend has been counted on. The last number word is the total.
>
> Finding an unknown addend (e.g., $8 + \square = 14$): One counts on from the known addend. The keeping track method is monitored so that counting on stops when the known total has been reached. The keeping track method tells the unknown addend.
>
> Subtracting ($14 - 8 = \square$): One thinks of subtracting as finding the unknown addend, as $8 + \square = 14$ and uses counting on to find an unknown addend (as above).

The problems in Table 2 which are solved by Level 1 methods in Kindergarten can also be solved using Level 2 methods: counting on to find the total (adding) or counting on to find the unknown addend (subtracting).

The middle difficulty (lightly shaded) problem types in Table 2 for Grade 1 are directly accessible with the embedded thinking of Level 2 methods and can be solved by counting on.

> Finding an unknown addend (e.g., $8 + \square = 14$) is used for Add To/Change Unknown, Put Together/Take Apart/Addend Unknown, and Compare/Difference Unknown. It is also used for Take From/Change Unknown ($14 - \square =$

> 8) after a student has decomposed the total into two addends, which means they can represent the situation as $14 - 8 = \square$.
>
> Adding or subtracting by counting on is used by some students for each of the kinds of Compare problems (see the equations in Table 2). Grade 1 students do not necessarily master the Compare Bigger Unknown or Smaller Unknown problems with the misleading language in the bottom row of Table 2.

Solving an equation such as $6 + 8 = \square$ by counting on from 8 relies on the understanding that $8 + 6$ gives the same total, an implicit use of the commutative property without the accompanying written representation $6 + 8 = 8 + 6$.

Level 3. Convert to an Easier Equivalent Problem.

Decompose an addend and compose a part with another addend.

These methods can be used to add or to find an unknown addend (and thus to subtract). These methods implicitly use the associative property.

Adding

Make a ten. E.g, for $8 + 6 = \square$,

$$8 + \underline{6} = 8 + \underline{2 + 4} = 10 + 4 = 14,$$

so $8 + 6$ becomes $10 + 4$.

Doubles plus or minus 1. E.g., for $6 + 7 = \square$,

$$6 + \underline{7} = 6 + \underline{6 + 1} = 12 + 1 = 13,$$

so $6 + 7$ becomes $12 + 1$.

Finding an unknown addend

Make a ten. E. g., for $8 + \square = 14$,

$$8 + \underline{2} = 10 \text{ and } \underline{4} \text{ more makes } 14.\ \underline{2 + 4} = 6.$$

So $8 + \square = 14$ is done as two steps: how many up to ten and how many over ten (which can be seen in the ones place of 14).

Doubles plus or minus 1. E.g., for $6 + \square = 13$,

$$6 + \underline{6 + 1} = 12 + 1.\ \underline{6 + 1} = 7.$$

So $6 + \square = 13$ is done as two steps: how many up to 12 ($6 + 6$) and how many from 12 to 13.

Subtracting

Thinking of subtracting as finding an unknown addend. E.g., solve $14 - 8 = \square$ or $13 - 6 = \square$ as $8 + \square = 14$ or $6 + \square = 13$ by the above methods (make a ten or doubles plus or minus 1).

Make a ten by going down over ten. E.g., $14 - 8 = \square$ can be done in two steps by going down over ten: $14 - 4$ (to get to 10) $- 4 = 6$.

The Level 1 and Level 2 problem types can be solved using these Level 3 methods.

Level 3 problem types can be solved by representing the situation with an equation or drawing, then re-representing to create a situation solved by adding, subtracting, or finding an unknown addend as shown above by methods at any level, but usually at Level 2 or 3. Many students only show in their writing part of this multi-step process of re-representing the situation.

> Students re-represent Add To/Start Unknown $\square + 6 = 14$ situations as $6 + \square = 14$ by using the commutative property (formally or informally).
>
> Students re-represent Take From/Start Unknown $\square - 8 = 6$ situations by reversing as $6 + 8 = \square$, which may then be solved by counting on from 8 or using a Level 3 method.

At Level 3, the Compare misleading language situations can be solved by representing the known quantities in a diagram that shows the bigger quantity in relation to the smaller quantity. The diagram allows the student to find a correct solution by representing the difference between quantities and seeing the relationship among the three quantities. Such diagrams are the same diagrams used for the other versions of compare situations; focusing on which quantity is bigger and which is smaller helps to overcome the misleading language.

Some students may solve Level 3 problem types by doing the above re-representing but use Level 2 counting on.

As students move through levels of solution methods, they increasingly use equations to represent problem situations as situation equations and then to re-represent the situation with a solution equation or a solution computation. They relate equations to diagrams, facilitating such re-representing. Labels on diagrams can help connect the parts of the diagram to the corresponding parts of the situation. But students may know and understand things that they may not use for a given solution of a problem as they increasingly do various representing and re-representing steps mentally.

Progressions for the Common Core State Standards in Mathematics (draft)

3 December 2012

High School, Algebra

Overview

Two domains in middle school are important in preparing students for Algebra in high school. In the progression in The Number System, students learn to see all numbers as part of a unified system, and become fluent in finding and using the properties of operations to find the values of numerical expressions that include those numbers. The Expressions and Equations Progression describes how students extend their use of these properties to linear equations and expressions with letters.

The Algebra category in high school is very closely allied with the Functions category:

- An expression in one variable can be viewed as defining a function: the act of evaluating the expression is an act of producing the function's output given the input.
- An equation in two variables can sometimes be viewed as defining a function, if one of the variables is designated as the input variable and the other as the output variable, and if there is just one output for each input. This is the case if the equation is in the form $y =$ (expression in x) or if it can be put into that form by solving for y.
- The notion of equivalent expressions can be understood in terms of functions: if two expressions are equivalent they define the same function.
- The solutions to an equation in one variable can be understood as the input values which yield the same output in the two functions defined by the expressions on each side of the equation. This insight allows for the method of finding approximate solutions by graphing the functions defined by each side and finding the points where the graphs intersect.

Because of these connections, some curricula take a functions-based approach to teaching algebra, in which functions are introduced early and used as a unifying theme for algebra. Other more traditional approaches introduce functions later, after extensive work with expressions and equations. The separation between Algebra and Functions in the standards is not intended to indicate a preference between these two approaches. It is, however, intended to specify the difference as mathematical concepts between expressions and equations on the one hand and functions on the other. Students often enter college-level mathematics courses with an apparent confusion between all three of these concepts. For example, when asked to factor a quadratic expression a student might instead find the solutions of the corresponding quadratic equation. Or another student might attempt to simplify the expression $\frac{\sin x}{x}$ by cancelling the x's.

The Algebra standards are fertile ground for the standards for mathematical practice. Two in particular that stand out are MP7, Look for and make use of structure, and MP8, Look for and express regularity in repeated reasoning. Students are expected to see how the structure of an algebraic expression reveals properties of the function it defines. They are expected to move from repeated reasoning with the slope formula to writing equations in various forms for straight lines, rather than memorizing all those forms separately. In this way the Algebra standards provide focus in a way different from the K–8 standards. Rather than focusing on a few topics, students in high school focus on a few seed ideas that lead to many different techniques.

Seeing Structure in Expressions

Students have been seeing expressions since Kindergarten, starting with arithmetic expressions in Grades K–5 and moving on to algebraic expressions in Grades 6–8. The middle grades standards in Expression and Equations build a ramp from arithmetic in elementary school to more sophisticated work with algebraic expression in high school. As the complexity of expressions increase, students continue to see them as being built out of basic operations: they see expressions as sums of terms and products of factors.[A-SSE.1a]

For example, in the example on the right, students compare $P+Q$ and $2P$ by seeing $2P$ as $P+P$. They distinguish between $(Q-P)/2$ and $Q-P/2$ by seeing the first as a quotient where the numerators is a difference and the second as a difference where the second term is a quotient. This last example also illustrates how students are able to see complicated expressions as built up out of simpler ones.[A-SSE.1b] As another example, students can see the expression $5+(x-1)^2$ as a sum of a constant and a square; and then see that inside the square term is the expression $x-1$. The first way of seeing tells them that it is always greater than or equal to 5, since a square is always greater than or equal to 0; the second way of seeing tells them that the square term is zero when $x=1$. Putting these together they can see that this expression attains its minimum value, 5, when $x=1$. The margin lists other tasks from the Illustrative Mathematics project (illustrativemathematics.org) for

Animal populations

Suppose P and Q give the sizes of two different animal populations, where $Q > P$. In 1–4, which of the given pair of expressions is larger? Briefly explain your reasoning in terms of the two populations.

1. $P+Q$ and $2P$
2. $\frac{P}{P+Q}$ and $\frac{P+Q}{2}$
3. $(Q-P)/2$ and $Q-P/2$
4. $P+50t$ and $Q+50t$

A-SSE.1a Interpret expressions that represent a quantity in terms of its context.

a Interpret parts of an expression, such as terms, factors, and coefficients.

A-SSE.1b Interpret expressions that represent a quantity in terms of its context.

b Interpret complicated expressions by viewing one or more of their parts as a single entity.

A-SSE.1.

Initially, the repertoire of operations for building up expressions is limited to the operations of arithmetic: addition, subtraction, multiplication and division (with the addition in middle grades of exponent notation to represent repeated multiplication). By the time they get to college, students have expanded that repertoire to include functions such as the square root function, exponential functions, and trigonometric functions.

For example, students in physics classes might be expected see the expression

$$L_0\sqrt{1-\frac{v^2}{c^2}},$$

which arises in the theory of special relativity, as the product of the constant L_0 and a term that is 1 when $v = 0$ and 0 when $v = c$—and furthermore, they might be expected to see this mentally, without having to go through a laborious process of evaluation. This involves combining large scale structure of the expression—a product of L_0 and another term—with the meaning of internal components such as $\frac{v^2}{c^2}$.

Seeing structure in expressions entails a dynamic view of an algebraic expression, in which potential rearrangements and manipulations are ever present.A-SSE.2 An important skill for college readiness is the ability to try out possible manipulations mentally without having to carry them out, and to see which ones might be fruitful and which not. For example, a student who can see

$$\frac{(2n+1)n(n+1)}{6}$$

as a polynomial in n with leading coefficient $\frac{1}{3}n^3$ has a leg up when it comes to calculus; a student who can mentally see the equivalence

$$\frac{R_1R_2}{R_1+R_2} = \frac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$$

without a laborious pencil and paper calculation is better equipped for a course in electrical engineering.

The standards avoid talking about simplification, because it is often not clear what the simplest form of an expression is, and even in cases where that is clear, it is not obvious that the simplest form is desirable for a given purpose. The standards emphasize purposeful transformation of expressions into equivalent forms that are suitable for the purpose at hand, as illustrated in the problem in the margin.A-SSE.3

For example, there are three commonly used forms for a quadratic expression:

- Standard form (e.g. $x^2 - 2x - 3$)
- Factored form (e.g. $(x+1)(x-3)$)

Illustrations of interpreting the structure of expression

The following tasks can be found by going to http://illustrativemathematics.org/illustrations/ and searching for A-SSE:

- Delivery Trucks
- Kitchen Floor Tiles
- Increasing or Decreasing? Variation 1
- Mixing Candies
- Mixing Fertilizer
- Quadrupling Leads to Halving
- The Bank Account
- The Physics Professor
- Throwing Horseshoes
- Animal Populations
- Equivalent Expressions
- Sum of Even and Odd

A-SSE.2 Use the structure of an expression to identify ways to rewrite it.

Which form is "simpler"?

A container of ice cream is taken from the freezer and sits in a room for t minutes. Its temperature in degrees Fahrenheit is $a - b \cdot 2^{-t} + b$, where a and b are positive constants.
Write this expression in a form that shows that the temperature is always

1. Less than $a + b$
2. Greater than a

The form $a + b - b \cdot 2^{-t}$ for the temperature shows that it is $a + b$ minus a positive number, so always less than $a + b$. On the other hand, the form $a + b(1 - 2^{-t})$ reveals that the temperature is always greater than a, cause it is a plus a positive number.

A-SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

- Vertex (or complete square) form (e.g. $(x-1)^2-4$).

Each is useful in different ways. The traditional emphasis on simplification as an automatic procedure might lead students to automatically convert the second two forms to the first, before considering which form is most useful in a given context.[A-SSE.3ab] This can lead to time consuming detours in algebraic work, such as solving $(x+1)(x-3)=0$ by first expanding and then applying the quadratic formula.

a Factor a quadratic expression to reveal the zeros of the function it defines.

b Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

The introduction of rational exponents and systematic practice with the properties of exponents in high school widen the field of operations for manipulating expressions.[A-SSE.3c] For example, students in later algebra courses who study exponential functions see

c Use the properties of exponents to transform expressions for exponential functions.

$$P(1+\frac{r}{12})^{12n} \quad \text{as} \quad P\left((1+\frac{r}{12})^{12}\right)^n$$

in order to understand formulas for compound interest.

Illustrations of writing expressions in equivalent forms

The following tasks can be found by going to http://illustrativemathematics.org/illustrations/ and searching for A-SSE:

- Ice Cream
- Increasing or Decreasing? Variation 2
- Profit of a company
- Seeing Dots

Much of the ability to see and use structure in transforming expressions comes from learning to recognize certain fundamental techniques. One such technique is recognizing internal cancellations, as in the expansion

$$(a-b)(a+b)=a^2-b^2.$$

An impressive example of this is

$$(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)=x^n-1,$$

in which all the terms cancel except the end terms. This identity is the foundation for the formula for the sum of a finite geometric series.[A-SSE.4]

A-SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.

Arithmetic with Polynomials and Rational Expressions

The development of polynomials and rational expressions in high school parallels the development of numbers in elementary school. In elementary school students might initially see expressions like $8+3$ and 11, or $\frac{3}{4}$ and 0.75, as fundamentally different: $8+3$ might be seen as describing a calculation and 11 is its answer; $\frac{3}{4}$ is a fraction and 0.75 is a decimal. Gradually they come to see numbers as forming a unified system, the number system, represented by points on the number line, and these different expressions are different ways of naming an underlying thing, a number.

A similar evolution takes place in algebra. At first algebraic expressions are simply numbers in which one or more letters are used to stand for a number which is either unspecified or unknown. Students learn to use the properties of operations to write expressions in different but equivalent forms. At some point they see equivalent expressions, particularly polynomial and rational expressions, as naming some underlying thing.[A-APR.1] There are at least two

A-APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

ways this can go. If the function concept is developed before or concurrently with the study of polynomials, then a polynomial can be identified with the function it defines. In this way $x^2 - 2x - 3$, $(x+1)(x-3)$, and $(x-1)^2 - 4$ are all the same polynomial because they all define the same function. Another approach is to think of polynomials as elements of a formal number system, in which you introduce the "number" x and see what numbers you can write down with it. Each approach has its advantages and disadvantages; the former approach is more common. Whichever is chosen, a curricular implementation might not necessarily explicitly state the choice, but should nonetheless be constructed in accordance with the implicit choice that has been made.

Either way, polynomials and rational expressions come to form a system in which things can be added, subtracted, multiplied and divided.[A-APR.7] Polynomials are analogous to the integers; rational expressions are analogous to the rational numbers.

A-APR.7 (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Polynomials form a rich ground for mathematical explorations that reveal relationships in the system of integers.[A-APR.4] For example, students can explore the sequence of squares

A-APR.4 Prove polynomial identities and use them to describe numerical relationships.

$$1, 4, 9, 16, 25, 36, \ldots$$

and notice that the differences between them—3, 5, 7, 9, 11—are consecutive odd integers. This mystery is explained by the polynomial identity

$$(n+1)^2 - n^2 = 2n + 1.$$

A more complex identity,

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2,$$

allows students to generate Pythagorean triples. For example, taking $x = 1$ and $y = 2$ in this identity yields $5^2 = 3^2 + 4^2$.

A particularly important polynomial identity, treated in advanced courses, is the Binomial Theorem[A-APR.5]

A-APR.5 (+) Know and apply the Binomial Theorem for the expansion of $(x+y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[1]

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + y^2,$$

for a positive integer n. The binomial coefficients can be obtained using Pascal's triangle

$$\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

in which each entry is the sum of the two above. Understanding why this rule follows algebraically from

$$(x+y)(x+y)^{n-1} = (x+y)^n$$

+ is excellent exercise in abstract reasoning (MP2) and in expressing
+ regularity in repeated reasoning (MP8).

Viewing polynomials as functions leads to explorations of a different nature. Polynomial functions are, on the one hand, very elementary, in that, unlike trigonometric and exponential functions, they are built up out of the basic operations of arithmetic. On the other hand, they turn out to be amazingly flexible, and can be used to approximate more advanced functions such as trigonometric and exponential functions. Although students only learn the complete story here if and when they study calculus, experience with constructing polynomial functions satisfying given conditions is useful preparation not only for calculus, but for understanding the mathematics behind curve-fitting methods used in applications to statistics and computer graphics.

A simple step in this direction is to construct polynomial functions with specified zeros.[A-APR.3] This is the first step in a progression which can lead, as an extension topic, to constructing polynomial functions whose graphs pass through any specified set of points in the plane.

A-APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

The analogy between polynomials and integers carries over to the idea of division with remainder. Just as in Grade 4 students find quotients and remainders of integers,[4.NBT.6] in high school they find quotients and remainders of polynomials.[A-APR.6] The method of polynomial long division is analogous to, and simpler than, the method of integer long division.

4.NBT.6 Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

A-APR.6 Rewrite simple rational expressions in different forms; write $\frac{a(x)}{b(x)}$ in the form $q(x) + \frac{r(x)}{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

A particularly important application of polynomial division is the case where a polynomial $p(x)$ is divided by a linear factor of the form $x - a$, for a real number a. In this case the remainder is the value $p(a)$ of the polynomial at $x = a$.[A-APR.2] It is a pity to see this topic reduced to "synthetic division," which reduced the method to a matter of carrying numbers between registers, something easily done by a computer, while obscuring the reasoning that makes the result evident. It is important to regard the Remainder Theorem as a theorem, not a techique.

A-APR.2 Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

A consequency of the Remainder Theorem is to establish the equivalence between linear factors and zeros that is the basis of much work with polynomials in high school: the fact that $p(a) = 0$ if and only if $x - a$ is a factor of $p(x)$. It is easy to see if $x - a$ is a factor then $p(a) = 0$. But the Remainder Theorem tells us that we can write

$$p(x) = (x - a)q(x) + p(a) \quad \text{for some polynomial } q(x).$$

In particular, if $p(a) = 0$ then $p(x) = (x - a)q(x)$, so $x - a$ is a factor of $p(x)$.

Creating Equations

Students have been writing equations, mostly linear equations, since middle grades. At first glance it might seem that the progression from middle grades to high school is fairly straightforward: the repertoire of functions that is acquired during high school allows students to create more complex equations, including equations arising from linear and quadratic functions, and simple rational and exponential functions;[A-CED.1] students are no longer limited largely to linear equations in modeling relationships between quantities with equations in two variables;[A-CED.2]; and students start to work with inequalities and systems of equations.[A-CED.3]

Two developments in high school complicate this picture. First, students in high school start using parameters in their equations, to represent whole classes of equations[F-LE.5] or to represent situations where the equation is to be adjusted to fit data.•

Second, modeling becomes a major objective in high school. Two of the standards just cited refer to "solving problems" and "interpreting solutions in a modeling context." And all the standards in the Creating Equations domain carry a modeling star, denoting their connection with the Modeling category in high school. This connotes not only an increase in the complexity of the equations studied, but an upgrade of the student's ability in every part of the modeling cycle, shown in the margin.

Variables, parameters, and constants Confusion about these terms plagues high school algebra. Here we try to set some rules for using them. These rules are not purely mathematical; indeed, from a strictly mathematical point of view there is no need for them at all. However, users of equations, by referring to letters as variables, parameters, or constants, can indicate how they intend to use the equations. This usage can be helpful if it is consistent.

In elementary and middle grades, life is easy. Elementary students solve problems with an unknown quantity, might use a symbol to stand for that quantity, and might call the symbol an unknown.[1.OA.2] In middle school students use variables systematically.[6.EE.6] They work with equations in one variable, such as $p + 0.05p = 10$ or equations in two variables such as $d = 5 + 5t$, relating two varying quantities.• In each case, apart from the variables, the numbers in the equation are given explicitly. The latter use presages the use of varibles to define functions.

In high school, things start to get complicated. For example, students consider the general equation for a straight line, $y = mx + b$. Here they are expected to understand that m and b are fixed for any given straight line, and that by varying m and b we obtain a whole family of straight lines. In this situation, m and b are called parameters. Of course, in an episode of mathematical work, the perspective could change; students might end up solving equations for m and b. Judging whether to explicitly indicate this—"now we

A-CED.1 Create equations and inequalities in one variable and use them to solve problems.

A-CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

A-CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

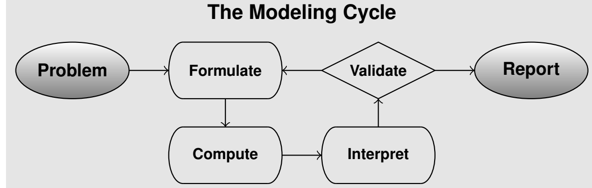
F-LE.5 Interpret the parameters in a linear or exponential function in terms of a context.

•

> Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems.

CCSSM, page 73

The Modeling Cycle



1.OA.2 Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

6.EE.6 Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

• Some writers prefer to retain the term unknown" for the first situation and the word "variable" for the second. This is not the usage adopted in the Standards.

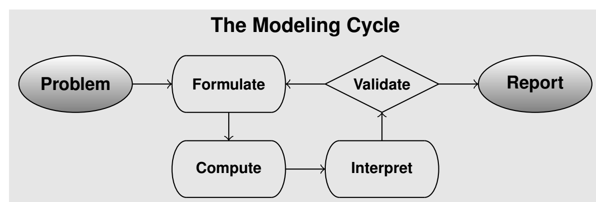
will regard the parameters as variables"—or whether to ignore it and just go ahead and solve for the parameters is a matter of pedagogical judgement.

Sometimes, and equation like $y = mx + b$ is used not to work with a parameterized family of equations but to consider the general form of an equation and prove something about it. For example, you might want take two points (x_1, y_1) and (x_2, y_2) on the graph of $y = mx + b$ and show that the slope between them is m. In this situation you might refer to m and b as constants rather than as parameters.

Finally, there are situations where an equation is used to describe the relationship between a number of different quantities, two which none of these terms apply.[A-CED.4] For example, Ohm's Law $V = IR$ relates the voltage, current, and resistance of an electrical circuit. An equation used in this way is sometimes called a formula. It is perhaps best to avoid entirely using the terms variable, parameter or constant when working with this formula, since there are 6 different ways it can be viewed as a defining one quantity as a function of the other with a third held constant.

A-CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

Different curricular implementations of the standards might navigate these terminological shoals differently (including trying to avoid them entirely).



Modeling with equations Consider the *Formulate* node in the modeling cycle. In elementary school students learn to formulat an equation to solve a word problem. For example, in solving

> Selina bought a shirt on sale that was 20% less than the original price. The original price was \$5 more than the sale price. What was the original price? Explain or show work.

students might let p be the original price in dollars and then express the sale price in terms of p in two different ways and set them equal. On the one hand the sale price is 20% less than the original price, and so equal to $p - 0.2p$. On the other hand it is \$5 less than the original price, and so equal to $p - 5$. Thus they want to solve the equation

$$p - 0.2p = p - 5.$$

In this task, he formulation of the equation tracks the text of the problem fairly closely, but requires more than a keyword reading of the text. For example, the second sentence needs to be reinterpreted as "the sale price is \$5 less than the original price." Since the words "less" and "more" are typically the subject of schemes for guessing the operation required in a problem without reading it, this shift is significant, and prepares students to read more difficult and realistic task statements.

Indeed, in a typical high school modeling problem, there might be significantly different ways of going about a problem depending on the choices made, and students must be much more strategic in formulating the model.

The *Compute* node of the modeling cycle is dealt with in the next section, on solving equations.

The *Interpret* node also becomes more complex. Equations in high school are also more likely to contain parameters than equations in earlier grades, and so interpreting a solution to an equation might involve more than consideration of a numerical value, but consideration of how the solution behaves as the parameters are varied.

The *Validate* node of the modeling cycle pulls together many of the standards for mathematical practice, including the modeling standard itself (MP4).

Reasoning with Equations and Inequalities

Equations in one variable

A naked equation, such as $x^2 = 4$, without any surrounding text, is merely a sentence fragment, neither true nor false, since it contains a variable x about which nothing is said. A written sequence of steps to solve an equation, such as in the margin, is code for a narrative line of reasoning using words like "if", "then", "for all" and "there exists." In the process of learning to solve equations, students learn certain standard "if-then" moves, for example "if $x = y$ then $x + 2 = y + 2$." The danger in learning algebra is that students emerge with nothing but the moves, which may make it difficult to detect incorrect or made-up moves later on. Thus the first requirement in the standards in this domain is that students understand that solving equations is a process of reasoning.[A-REI.1] This does not necessarily mean that they always write out the full text; part of the advantage of algebraic notation is its compactness. Once students know what the code stands for, they can start writing in code. Thus, eventually students might make $x^2 = 4 \implies x = \pm 2$ one step.[2]

Understanding solving equations as a process of reasoning demystifies "extraneous" solutions that can arise under certain solution procedures.[A-REI.2] The flow of reasoning is forward, from the assumption that a number x satisfies the equation to a list of possibilites for x. But not all the steps are necessarily reversible, and so it is not necessarily true that every number in the list satisfies the equation. For example, it is true that if $x = 2$ then $x^2 = 4$. But it is not true that if $x^2 = 4$ then $x = 2$ (it might be that $x = -2$). Squaring both sides of an equation is a typical example of an irreversible step; another is multiplying both sides of the equation by a quantity that might be zero (see margin for examples).

With an understanding of solving equations as a reasoning process, students can organize the various methods for solving different

[2]It should be noted, however, that calling this step "taking the square root of both sides" is dangerous, since it leads to the erroneous belief that $\sqrt{4} = \pm 2$.

Fragments of reasoning

$$\begin{aligned} x^2 &= 4 \\ x^2 - 4 &= 0 \\ (x-2)(x+2) &= 0 \\ x &= 2, -2 \end{aligned}$$

This sequence of equations is short-hand for a line of reasoning: "If x is a number whose square is 4, then $x^2 - 4 = 0$. Since $x^2 - 4 = (x-2)(x+2)$ for all numbers x, it follows that $(x-2)(x+2) = 0$. So either $x - 2 = 0$, in which case $x = 2$, or $x + 2 = 0$, in which case $x = -2$." More might be said: a justification of the last step, for example, or a check that 2 and -2 actually do satisfy the equation, which has not been proved by this line of reasoning.

A-REI.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

A-REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

types of equations into a coherent picture. For example, solving linear equations involves only steps that are reversible (adding a constant to both sides, multiplying both sides by a non-zero constant, transforming an expression on one side into an equivalent expression). Therefore solving linear equations does not produce extraneous solutions.[A-REI.3] The process of completing the square also involves only this same list of steps, and so converts any quadratic equation into an equivalent equation of the form $(x-p)^2 = q$ that has exactly the same solutions[A-REI.4a] The latter equation is easy to solve by the reasoning explained above.

This example sets up a theme that reoccurs throughout algebra; finding ways of transforming equations into certain standard forms that have the same solutions. For example, any exponential equation can be transformed into the form $b^x = a$, the solution to which is (by definition) a logarithm.

It is traditional for students to spend a lot of time on various techniques of solving quadratic equations, which are often presented as if they are completely unrelated (factoring, completing the square, the quadratic formula). In fact, as we have seen, the key step in completing the square, going from $x^2 = q$ to $x = \pm\sqrt{q}$, involves at its heart factoring. And the quadratic formula is nothing more than an encapsulation of the method of completing the square. Rather than long drills on techniques of dubious value, students with an understanding of the underlying reasoning behind all these methods are opportunistic in their application, choosing the method that bests suits the situation at hand.[A-REI.4b]

Systems of equations

Student work with solving systems of equations starts the same way as work with solving equations in one variable; with an understanding of the reasoning behind the various techniques.[A-REI.5] An important step is realizing that a solution to a system of equations must be a solution all of the equations in the system simultaneously. Then the process of adding one equation to another is understood as "if the two sides of one equation are equal, and the two sides of another equation are equal, then the sum of the left sides of the two equations is equal to the sum of the right sides." Since this reasoning applies equally to subtraction, the process of adding one equation to another is reversible, and therefore leads to an equivalent system of equations.

Understanding these points for the particular case of two equations in two variables is preparation for more general situations. Such systems also have the advantage that a good graphical visualization is available; a pair (x, y) satisfies two equations in two variables if it is on both their graphs, and therefore an intersection point of the graphs.[A-REI.6]

Another important method of solving systems is the method of substitution. Again this can be understood in terms of simultaneity;

A-REI.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

A-REI.4a Solve quadratic equations in one variable.

a Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x-p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

b Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

A-REI.5 Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

A-REI.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

if (x, y) satisfies two equations simultaneously, then the expression for y in terms of x obtained from the first equation should form a true statement when substituted into the second equation. Since a linear equation can always be solved for one of the variables in it, this is a good method when just one of the equations in a system is linear. [A-REI.7]

In more advanced courses, students see systems of linear equations in many variables as single matrix equations in vector variables.[A-REI.8, A-REI.9]

A-REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.

A-REI.8 (+) Represent a system of linear equations as a single matrix equation in a vector variable.

A-REI.9 (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater).

Visualizing solutions graphically

Just as the algebraic work with equations can be reduced to a series of algebraic moves unsupported by reasoning, so can the graphical visualization of solutions. The simple idea that an equation $f(x) = g(x)$ can be solved (approximately) by graphing $y = f(x)$ and $y = g(x)$ and finding the intersection points involves a number of pieces of conceptual understanding.[A-REI.11] This seemingly simple method, often treated as obvious, involves the rather sophisticated move of reversing the reduction of an equation in two variables to an equation in one variable. Rather, it seeks to convert an equation in one variable, $f(x) = g(x)$, to a system of equations in two variables, $y = f(x)$ and $y = g(x)$, by introducing a second variable y and setting it equal to each side of the equation. If x is a solution to the original equation then $f(x)$ and $g(x)$ are equal, and thus (x, y) is a solution to the new system. This reasoning is often tremendously compressed and presented as obvious graphically; in fact following it graphically in a specific example can be instructive. [Give example in margin.]

A-REI.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

Fundamental to all of this is a simple understanding of what a graph of an equation in two variables means.[A-REI.10]

A-REI.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

Progressions for the Common Core State Standards in Mathematics (draft)*

12 August 2011

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[0]This document can be read with Preview on a Mac or with the latest version of Adobe Acrobat on a Mac or PC. It does not work with earlier versions of Acrobat.

3–5 Number and Operations—Fractions

Overview

Overview to be written.

Note. Changes such as including relevant equations or replacing with tape diagrams or fraction strips are planned for some diagrams. Some readers may find it helpful to create their own equations or representations.

Grade 3

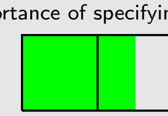
The meaning of fractions In Grades 1 and 2, students use fraction language to describe partitions of shapes into equal shares.[2.G.3] In Grade 3 they start to develop the idea of a fraction more formally, building on the idea of partitioning a whole into equal parts. The whole can be a shape such as a circle or rectangle, a line segment, or any one finite entity susceptible to subdivision and measurement. In Grade 4, this is extended to include wholes that are collections of objects.

[2.G.3] Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

Grade 3 students start with unit fractions (fractions with numerator 1), which are formed by partitioning a whole into equal parts and taking one part, e.g., if a whole is partitioned into 4 equal parts then each part is $\frac{1}{4}$ of the whole, and 4 copies of that part make the whole. Next, students build fractions from unit fractions, seeing the numerator 3 of $\frac{3}{4}$ as saying that $\frac{3}{4}$ is the quantity you get by putting 3 of the $\frac{1}{4}$'s together.[3.NF.1] They read any fraction this way, and in particular there is no need to introduce "proper fractions" and "improper fractions" initially; $\frac{5}{3}$ is the quantity you get by combining 5 parts together when the whole is divided into 3 equal parts.

[3.NF.1] Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.

The importance of specifying the whole



Without specifying the whole it is not reasonable to ask what fraction is represented by the shaded area. If the left square is the whole, the shaded area represents the fraction $\frac{3}{2}$; if the entire rectangle is the whole, the shaded area represents $\frac{3}{4}$.

Two important aspects of fractions provide opportunities for the mathematical practice of attending to precision (MP6):

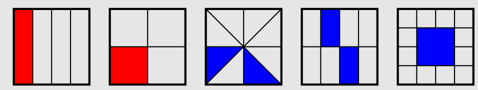
- Specifying the whole.
- Explaining what is meant by "equal parts."

Initially, students can use an intuitive notion of congruence ("same size and same shape") to explain why the parts are equal, e.g., when they divide a square into four equal squares or four equal rectangles.

Students come to understand a more precise meaning for "equal parts" as "parts with equal measurements." For example, when a ruler is partitioned into halves or quarters of an inch, they see that each subdivision has the same length. In area models they reason about the area of a shaded region to decide what fraction of the whole it represents (MP3).

The goal is for students to see unit fractions as the basic building blocks of fractions, in the same sense that the number 1 is the basic building block of the whole numbers; just as every whole number is obtained by combining a sufficient number of 1s, every fraction is obtained by combining a sufficient number of unit fractions.

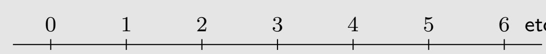
Area representations of $\frac{1}{4}$



In each representation the square is the whole. The two squares on the left are divided into four parts that have the same size and shape, and so the same area. In the three squares on the right, the shaded area is $\frac{1}{4}$ of the whole area, even though it is not easily seen as one part in a division of the square into four parts of the same shape and size.

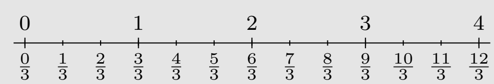
The number line and number line diagrams On the number line, the whole is the *unit interval*, that is, the interval from 0 to 1, measured by length. Iterating this whole to the right marks off the whole numbers, so that the intervals between consecutive whole numbers, from 0 to 1, 1 to 2, 2 to 3, etc., are all of the same length, as shown. Students might think of the number line as an infinite ruler.

The number line



To construct a unit fraction on a number line diagram, e.g. $\frac{1}{3}$, students partition the unit interval into 3 intervals of equal length and recognize that each has length $\frac{1}{3}$. They locate the number $\frac{1}{3}$ on the number line by marking off this length from 0, and locate other fractions with denominator 3 by marking off the number of lengths indicated by the numerator.[3.NF.2]

The number line marked off in thirds



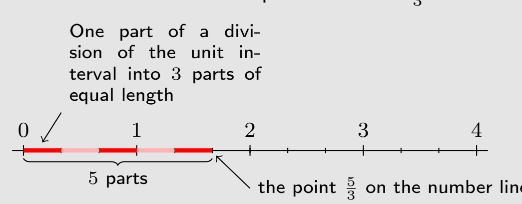
3.NF.2 Understand a fraction as a number on the number line; represent fractions on a number line diagram.

- a Represent a fraction $1/b$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $1/b$ and that the endpoint of the part based at 0 locates the number $1/b$ on the number line.
- b Represent a fraction a/b on a number line diagram by marking off a lengths $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.

Students sometimes have difficulty perceiving the unit on a number line diagram. When locating a fraction on a number line diagram, they might use as the unit the entire portion of the number line that is shown on the diagram, for example indicating the number 3 when asked to show $\frac{3}{4}$ on a number line diagram marked from 0 to 4. Although number line diagrams are important representations for students as they develop an understanding of a fraction as a number, in the early stages of the NF Progression they use other representations such as area models, tape diagrams, and strips of paper. These, like number line diagrams, can be subdivided, representing an important aspect of fractions.

The number line reinforces the analogy between fractions and whole numbers. Just as 5 is the point on the number line reached by marking off 5 times the length of the unit interval from 0, so $\frac{5}{3}$ is the point obtained in the same way using a different interval as the basic unit of length, namely the interval from 0 to $\frac{1}{3}$.

Number line representation of $\frac{5}{3}$



Equivalent fractions Grade 3 students do some preliminary reasoning about equivalent fractions, in preparation for work in Grade 4. As students experiment on number line diagrams they discover that many fractions label the same point on the number line, and are therefore equal; that is, they are *equivalent fractions*. For example, the fraction $\frac{1}{2}$ is equal to $\frac{2}{4}$ and also to $\frac{3}{6}$. Students can also use fraction strips to see fraction equivalence.[3.NF.3ab]

In particular, students in Grade 3 see whole numbers as fractions, recognizing, for example, that the point on the number line designated by 2 is now also designated by $\frac{2}{1}$, $\frac{4}{2}$, $\frac{6}{3}$, $\frac{8}{4}$, etc. so that[3.NF.3c]

$$2 = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \cdots .$$

Of particular importance are the ways of writing 1 as a fraction:

$$1 = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5} = \cdots .$$

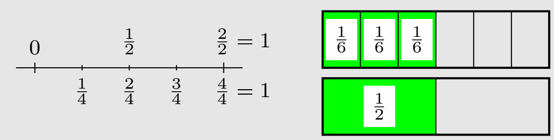
3.NF.3abcExplain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

a Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.

b Recognize and generate simple equivalent fractions, e.g., $1/2 = 2/4$, $4/6 = 2/3$. Explain why the fractions are equivalent, e.g., by using a visual fraction model.

c Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers.

Using the number line and fraction strips to see fraction equivalence



Comparing fractions Previously, in Grade 2, students compared lengths using a standard measurement unit.[2.MD.3] In Grade 3 they build on this idea to compare fractions with the same denominator. They see that for fractions that have the same denominator, the underlying unit fractions are the same size, so the fraction with the greater numerator is greater because it is made of more unit fractions. For example, segment from 0 to $\frac{3}{4}$ is shorter than the segment from 0 to $\frac{5}{4}$ because it measures 3 units of $\frac{1}{4}$ as opposed to 5 units of $\frac{1}{4}$. Therefore $\frac{3}{4} < \frac{5}{4}$.[3.NF.3d]

Students also see that for unit fractions, the one with the larger denominator is smaller, by reasoning, for example, that in order for more (identical) pieces to make the same whole, the pieces must be smaller. From this they reason that for fractions that have the same numerator, the fraction with the smaller denominator is greater. For example, $\frac{2}{5} > \frac{2}{7}$, because $\frac{1}{7} < \frac{1}{5}$, so 2 lengths of $\frac{1}{7}$ is less than 2 lengths of $\frac{1}{5}$.

As with equivalence of fractions, it is important in comparing fractions to make sure that each fraction refers to the same whole.

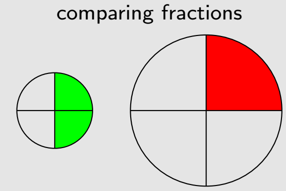
As students move towards thinking of fractions as points on the number line, they develop an understanding of order in terms of position. Given two fractions—thus two points on the number line—the one to the left is said to be smaller and the one to right is said to be larger. This understanding of order as position will become important in Grade 6 when students start working with negative numbers.

2.MD.3Estimate lengths using units of inches, feet, centimeters, and meters.

3.NF.3d Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

d Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual fraction model.

The importance of referring to the same whole when comparing fractions



A student might think that $\frac{1}{4} > \frac{1}{2}$, because a fourth of the pizza on the right is bigger than a half of the pizza on the left.

Grade 4

Grade 4 students learn a fundamental property of equivalent fractions: multiplying the numerator and denominator of a fraction by the same non-zero whole number results in a fraction that represents the same number as the original fraction. This property forms the basis for much of their other work in Grade 4, including the comparison, addition, and subtraction of fractions and the introduction of finite decimals.

Equivalent fractions Students can use area models and number line diagrams to reason about equivalence.[4.NF.1] They see that the numerical process of multiplying the numerator and denominator of a fraction by the same number, *n*, corresponds physically to partitioning each unit fraction piece into *n* smaller equal pieces. The whole is then partitioned into *n* times as many pieces, and there are *n* times as many smaller unit fraction pieces as in the original fraction.

This argument, once understood for a range of examples, can be seen as a general argument, working directly from the Grade 3 understanding of a fraction as a point on the number line.

The fundamental property can be presented in terms of division, as in, e.g.

$$\frac{28}{36} = \frac{28 \div 4}{36 \div 4} = \frac{7}{9}.$$

Because the equations $28 \div 4 = 7$ and $36 \div 4 = 9$ tell us that $28 = 4 \times 7$ and $36 = 4 \times 9$, this is the fundamental fact in disguise:

$$\frac{4 \times 7}{4 \times 9} = \frac{7}{9}.$$

It is possible to over-emphasize the importance of simplifying fractions in this way. There is no mathematical reason why fractions must be written in simplified form, although it may be convenient to do so in some cases.

Grade 4 students use their understanding of equivalent fractions to compare fractions with different numerators and different denominators.[4.NF.2] For example, to compare $\frac{5}{8}$ and $\frac{7}{12}$ they rewrite both fractions as

$$\frac{60}{96}\left(= \frac{12 \times 5}{12 \times 8}\right) \quad \text{and} \quad \frac{56}{96}\left(= \frac{7 \times 8}{12 \times 8}\right)$$

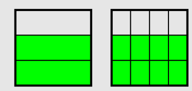
Because $\frac{60}{96}$ and $\frac{56}{96}$ have the same denominator, students can compare them using Grade 3 methods and see that $\frac{56}{96}$ is smaller, so

$$\frac{7}{12} < \frac{5}{8}.$$

Students also reason using benchmarks such as $\frac{1}{2}$ and 1. For example, they see that $\frac{7}{8} < \frac{13}{12}$ because $\frac{7}{8}$ is less than 1 (and is

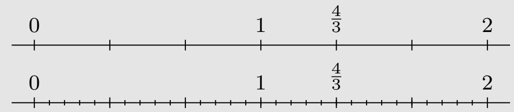
4.NF.1 Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

Using an area model to show that $\frac{2}{3} = \frac{4\times 2}{4\times 3}$



The whole is the square, measured by area. On the left it is divided horizontally into 3 *rectangles of equal area, and the shaded region is* 2 *of these and so represents* $\frac{2}{3}$*. On the right it is divided into* 4×3 *small rectangles of equal area, and the shaded area comprises* 4×2 *of these, and so it represents* $\frac{4\times 2}{4\times 3}$*.*

Using the number line to show that $\frac{4}{3} = \frac{5\times 4}{5\times 3}$



$\frac{4}{3}$ *is 4 parts when each part is* $\frac{1}{3}$*, and we want to see that this is also* 5×4 *parts when each part is* $\frac{1}{5\times 3}$*. Divide each of the intervals of length* $\frac{1}{3}$ *into 5 parts of equal length. There are* 5×3 *parts of equal length in the unit interval, and* $\frac{4}{3}$ *is* 5×4 *of these. Therefore* $\frac{4}{3} = \frac{5\times 4}{5\times 3} = \frac{20}{15}$*.*

4.NF.2 Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual fraction model.

therefore to the left of 1) but $\frac{13}{12}$ is greater than 1 (and is therefore to the right of 1).

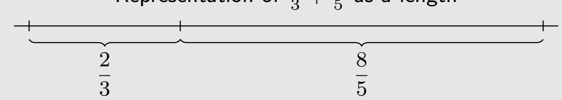
Grade 5 students who have learned about fraction multiplication can see equivalence as "multiplying by 1":

$$\frac{7}{9} = \frac{7}{9} \times 1 = \frac{7}{9} \times \frac{4}{4} = \frac{28}{36}$$

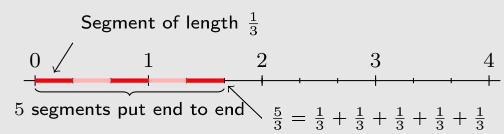
However, although a useful mnemonic device, this does not constitute a valid argument at this grade, since students have not yet learned fraction multiplication.

Adding and subtracting fractions The meaning of addition is the same for both fractions and whole numbers, even though algorithms for calculating their sums can be different. Just as the sum of 4 and 7 can be seen as the length of the segment obtained by joining together two segments of lengths 4 and 7, so the sum of $\frac{2}{3}$ and $\frac{8}{5}$ can be seen as the length of the segment obtained joining together two segments of length $\frac{2}{3}$ and $\frac{8}{5}$. It is not necessary to know how much $\frac{2}{3} + \frac{8}{5}$ is exactly in order to know what the sum means. This is analogous to understanding 51×78 as 51 groups of 78, without necessarily knowing its exact value.

Representation of $\frac{2}{3} + \frac{8}{5}$ as a length



Using the number line to see that $\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$



This simple understanding of addition as putting together allows students to see in a new light the way fractions are built up from unit fractions. The same representation that students used in Grade 4 to see a fraction as a point on the number line now allows them to see a fraction as a sum of unit fractions: just as $5 = 1+1+1+1+1$, so

$$\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

because $\frac{5}{3}$ is the total length of 5 copies of $\frac{1}{3}$.[4.NF.3]

Armed with this insight, students decompose and compose fractions with the same denominator. They add fractions with the same denominator:[4.NF.3c]

$$\begin{aligned} \frac{7}{5} + \frac{4}{5} &= \overbrace{\frac{1}{5} + \cdots \frac{1}{5}}^{7} + \overbrace{\frac{1}{5} + \cdots \frac{1}{5}}^{4} \\ &= \frac{\overbrace{1 + 1 + \cdots + 1}^{7+4}}{5} \\ &= \frac{7+4}{5}. \end{aligned}$$

Using the understanding gained from work with whole numbers of the relationship between addition and subtraction, they also subtract fractions with the same denominator. For example, to subtract $\frac{5}{6}$ from $\frac{17}{6}$, they decompose

$$\frac{17}{6} = \frac{12}{6} + \frac{5}{6}, \quad \text{so} \quad \frac{17}{6} - \frac{5}{6} = \frac{17-5}{6} = \frac{12}{6} = 2.$$

4.NF.3 Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

a Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.

b Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model.

c Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.

Students also compute sums of whole numbers and fractions, by representing the whole number as an equivalent fraction with the same denominator as the fraction, e.g.

$$7\frac{1}{5} = 7 + \frac{1}{5} = \frac{35}{5} + \frac{1}{5} = \frac{36}{5}.$$

Students use this method to add mixed numbers with like denominators.• Converting a mixed number to a fraction should not be viewed as a separate technique to be learned by rote, but simply as a case of fraction addition.

Similarly, converting an improper fraction to a mixed number is a matter of decomposing the fraction into a sum of a whole number and a number less than 1.[4.NF.3b] Students can draw on their knowledge from Grade 3 of whole numbers as fractions. For example, knowing that $1 = \frac{3}{3}$, they see

$$\frac{5}{3} = \frac{3}{3} + \frac{2}{3} = 1 + \frac{2}{3} = 1\frac{2}{3}.$$

Repeated reasoning with examples that gain in complexity leads to a general method involving the Grade 4 NBT skill of finding quotients and remainders.[4.NBT.6] For example,

$$\frac{47}{6} = \frac{(7 \times 6) + 5}{6} = \frac{7 \times 6}{6} + \frac{5}{6} = 7 + \frac{5}{6} = 7\frac{5}{6}.$$

When solving word problems students learn to attend carefully to the underlying unit quantities. In order to formulate an equation of the form $A+B=C$ or $A-B=C$ for a word problem, the numbers A, B, and C must all refer to the same (or equivalent) wholes or unit amounts.[4.NF.3d] For example, students understand that the problem

> Bill had $\frac{2}{3}$ of a cup of juice. He drank $\frac{1}{2}$ of his juice. How much juice did Bill have left?

cannot be solved by subtracting $\frac{2}{3} - \frac{1}{2}$ because the $\frac{2}{3}$ refers to a cup of juice, but the $\frac{1}{2}$ refers to the amount of juice that Bill had, and not to a cup of juice. Similarly, in solving

> If $\frac{1}{4}$ of a garden is planted with daffodils, $\frac{1}{3}$ with tulips, and the rest with vegetables, what fraction of the garden is planted with flowers?

students understand that the sum $\frac{1}{3} + \frac{1}{4}$ tells them the fraction of the garden that was planted with flowers, but not the number of flowers that were planted.

• A mixed number is a whole number plus a fraction smaller than 1, written without the $+$ sign, e.g. $5\frac{3}{4}$ means $5 + \frac{3}{4}$ and $7\frac{1}{5}$ means $7 + \frac{1}{5}$.

4.NF.3b Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

b Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model.

4.NBT.6 Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

4.NF.3d Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

d Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem.

Multiplication of a fraction by a whole number Previously in Grade 3, students learned that 3×7 can be represented as the number of objects in 3 groups of 7 objects, and write this as $7 + 7 + 7$. Grade 4 students apply this understanding to fractions, seeing

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \quad \text{as} \quad 5 \times \frac{1}{3}.$$

In general, they see a fraction as the numerator times the unit fraction with the same denominator,[4.NF.4a] e.g.,

$$\frac{7}{5} = 7 \times \frac{1}{5}, \qquad \frac{11}{3} = 11 \times \frac{1}{3}.$$

The same thinking, based on the analogy between fractions and whole numbers, allows students to give meaning to the product of a whole number and a fraction,[4.NF.4b] e.g., they see

$$3 \times \frac{2}{5} \quad \text{as} \quad \frac{2}{5} + \frac{2}{5} + \frac{2}{5} = \frac{3 \times 2}{5} = \frac{6}{5}.$$

Students solve word problems involving multiplication of a fraction by a whole number.[4.NF.4c]

> If a bucket holds $2\frac{3}{4}$ gallons and 43 buckets of water fill a tank, how much does the tank hold?

The answer is $43 \times 2\frac{3}{4}$ gallons, which is

$$43 \times \left(2 + \frac{3}{4}\right) = 43 \times \frac{11}{4} = \frac{473}{4} = 118\frac{1}{4} \text{ gallons}$$

Decimals Fractions with denominator 10 and 100, called *decimal fractions*, arise naturally when student convert between dollars and cents, and have a more fundamental importance, developed in Grade 5, in the base 10 system. For example, because there are 10 dimes in a dollar, 3 dimes is $\frac{3}{10}$ of a dollar; and it is also $\frac{30}{100}$ of a dollar because it is 30 cents, and there are 100 cents in a dollar. Such reasoning provides a concrete context for the fraction equivalence

$$\frac{3}{10} = \frac{3 \times 10}{10 \times 10} = \frac{30}{100}.$$

Grade 3 students learn to add decimal fractions by converting them to fractions with the same denominator, in preparation for general fraction addition in Grade 5:[4.NF.5]

$$\frac{3}{10} + \frac{27}{100} = \frac{30}{100} + \frac{27}{100} = \frac{57}{100}.$$

They can interpret this as saying that 3 dimes together with 27 cents make 57 cents.

Fractions with denominators equal to 10, 100, etc., such

$$\frac{27}{10}, \quad \frac{27}{100}, \quad \text{etc.}$$

can be written by using a *decimal point* as[4.NF.6]

$$2.7, \quad 0.27.$$

The number of digits to the right of the decimal point indicates the number of zeros in the denominator, so that $2.70 = \frac{270}{100}$ and

4.NF.4 Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

a Understand a fraction a/b as a multiple of $1/b$.

b Understand a multiple of a/b as a multiple of $1/b$, and use this understanding to multiply a fraction by a whole number.

c Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem.

4.NF.5 Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.[1]

4.NF.6 Use decimal notation for fractions with denominators 10 or 100.

$2.7 = \frac{27}{10}$. Students use their ability to convert fractions to reason that $2.70 = 2.7$ because

$$2.70 = \frac{270}{100} = \frac{10 \times 27}{10 \times 10} = \frac{27}{10} = 2.7.$$

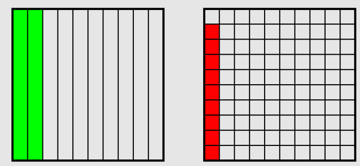
Students compare decimals using the meaning of a decimal as a fraction, making sure to compare fractions with the same denominator. For example, to compare 0.2 and 0.09, students think of them as 0.20 and 0.09 and see that $0.20 > 0.09$ because[4.NF.7]

$$\frac{20}{100} > \frac{9}{100}.$$

The argument using the meaning of a decimal as a fraction generalizes to work with decimals in Grade 5 that have more than two digits, whereas the argument using a visual fraction model, shown in the margin, does not. So it is useful for Grade 4 students to see such reasoning.

4.NF.7 Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual model.

Seeing that $0.2 > 0.09$ using a visual fraction model



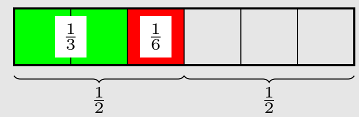
The shaded region on the left shows 0.2 *of the unit square, since it is two parts when the square is divided int 10 parts of equal area. The shaded region on the right shows* 0.09 *of the unit square, since it is 9 parts when the unit is divided into 100 parts of equal area.*

Grade 5

Adding and subtracting fractions In Grade 4, students calculate sums of fractions with different denominators where one denominator is a divisor of the other, so that only one fraction has to be changed. For example, they might have used a fraction strip to reason that

$$\frac{1}{3}+\frac{1}{6}=\frac{2}{6}+\frac{1}{6}=\frac{3}{6}=\frac{1}{2},$$

Using a fraction strip to show that $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$



and in working with decimals they added fractions with denominators 10 and 100, such as

$$\frac{2}{10}+\frac{7}{100}=\frac{20}{100}+\frac{7}{100}=\frac{27}{100}.$$

They understand the process as expressing both summands in terms of the same unit fraction so that they can be added. Grade 5 students extend this reasoning to situations where it is necessary to re-express both fractions in terms of a new denominator.[5.NF.1] For example, in calculating $\frac{2}{3}+\frac{5}{4}$ they reason that if each third in $\frac{2}{3}$ is subdivided into fourths, and if each fourth in $\frac{5}{4}$ is subdivided into thirds, then each fraction will be a sum of unit fractions with denominator $3\times 4=4\times 3=12$:

5.NF.1 Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators.

$$\frac{2}{3}+\frac{5}{4}=\frac{2\times 4}{3\times 4}+\frac{5\times 3}{4\times 3}=\frac{8}{12}+\frac{15}{12}=\frac{23}{12}.$$

In general two fractions can be added by subdividing the unit fractions in one using the denominator of the other:

$$\frac{a}{b}+\frac{c}{d}=\frac{a\times d}{b\times d}+\frac{c\times b}{d\times b}=\frac{ad+bc}{bd}.$$

It is not necessary to find a least common denominator to calculate sums of fractions, and in fact the effort of finding a least common denominator is a distraction from understanding algorithms for adding fractions.

Students make sense of fractional quantities when solving word problems, estimating answers mentally to see if they make sense.[5.NF.2] For example in the problem

5.NF.2 Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers.

> Ludmilla and Lazarus each have a lemon. They need a cup of lemon juice to make hummus for a party. Ludmilla squeezes $\frac{1}{2}$ a cup from hers and Lazarus squeezes $\frac{2}{5}$ of a cup from his. How much lemon juice to they have? Is it enough?

students estimate that there is almost but not quite one cup of lemon juice, because $\frac{2}{5}<\frac{1}{2}$. They calculate $\frac{1}{2}+\frac{2}{5}=\frac{9}{10}$, and see this as $\frac{1}{10}$ less than 1, which is probably a small enough shortfall that it will not ruin the recipe. They detect an incorrect result such as $\frac{2}{5}+\frac{2}{5}=\frac{3}{7}$ by noticing that $\frac{3}{7}<\frac{1}{2}$.

Multiplying and dividing fractions In Grade 4 students connected fractions with addition and multiplication, understanding that

$$\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 5 \times \frac{1}{3}.$$

In Grade 5, they connect fractions with division, understanding that

$$5 \div 3 = \frac{5}{3},$$

or, more generally, $\frac{a}{b} = a \div b$ for whole numbers a and b, with b not equal to zero.[5.NF.3] They can explain this by working with their understanding of division as equal sharing (see figure in margin). They also create story contexts to represent problems involving division of whole numbers. For example, they see that

> If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get?

can be solved in two ways. First, they might partition each pound among the 9 people, so that each person gets $50 \times \frac{1}{9} = \frac{50}{9}$ pounds. Second, they might use the equation $9 \times 5 = 45$ to see that each person can be given 5 pounds, with 5 pounds remaining. Partitioning the remainder gives $5\frac{5}{9}$ pounds for each person.

Students have, since Grade 1, been using language such as "third of" to describe one part when a whole is partitioned into three parts. With their new understanding of the connection between fractions and division, students now see that $\frac{5}{3}$ is one third of 5, which leads to the meaning of multiplication by a unit fraction:

$$\frac{1}{3} \times 5 = \frac{5}{3}.$$

This in turn extends to multiplication of any quantity by a fraction.[5.NF.4a] Just as

$$\frac{1}{3} \times 5 \text{ is one part when } 5 \text{ is partitioned into } 3 \text{ parts,}$$

so

$$\frac{4}{3} \times 5 \text{ is 4 parts when } 5 \text{ is partitioned into } 3 \text{ parts.}$$

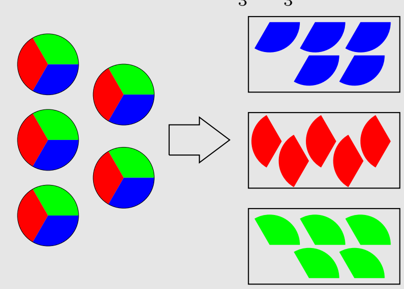
Using this understanding of multiplication by a fraction, students develop the general formula for the product of two fractions,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd},$$

for whole numbers a, b, c, d, with b, d not zero. Grade 5 students need not express the formula in this general algebraic form, but rather reason out many examples using fraction strips and number line diagrams.

5.NF.3 Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

How to share 5 objects equally among 3 shares:
$5 \div 3 = 5 \times \frac{1}{3} = \frac{5}{3}$

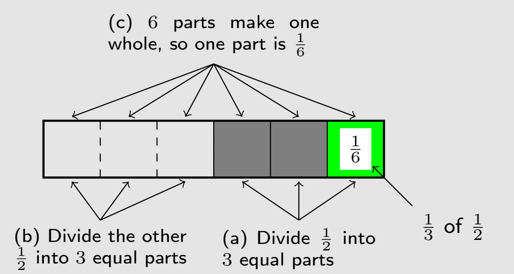


If you divide 5 objects equally among 3 shares, each of the 5 objects should contribute $\frac{1}{3}$ of itself to each share. Thus each share consists of 5 pieces, each of which is $\frac{1}{3}$ of an object, and so each share is $5 \times \frac{1}{3} = \frac{5}{3}$ of an object.

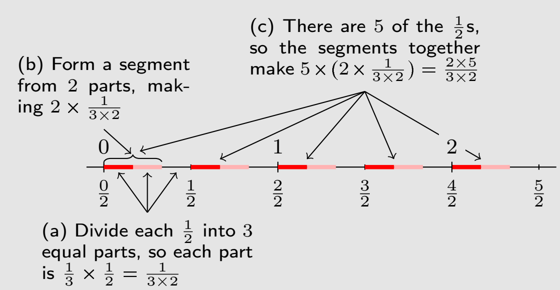
5.NF.4a Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

a Interpret the product $(a/b) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$.

Using a fraction strip to show that $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$



Using a number line to show that $\frac{2}{3} \times \frac{5}{2} = \frac{2 \times 5}{3 \times 2}$



For more complicated examples, an area model is useful, in which students work with a rectangle that has fractional side lengths, dividing it up into rectangles whose sides are the corresponding unit fractions.

Students also understand fraction multiplication by creating story contexts. For example, to explain

$$\frac{2}{3} \times 4 = \frac{8}{3},$$

they might say

> Ron and Hermione have 4 pounds of Bertie Bott's Every Flavour Beans. They decide to share them 3 ways, saving one share for Harry. How many pounds of beans do Ron and Hermione get?

Using the relationship between division and multiplication, students start working with simple fraction division problems. Having seen that division of a whole number by a whole number, e.g. $5 \div 3$, is the same as multiplying the number by a unit fraction, $\frac{1}{3} \times 5$, they now extend the same reasoning to division of a unit fraction by a whole number, seeing for example that[5.NF.7a]

$$\frac{1}{6} \div 3 = \frac{1}{6 \times 3} = \frac{1}{18}.$$

Also, they reason that since there are 6 portions of $\frac{1}{6}$ in 1, there must be 3×6 in 3, and so[5.NF.7b]

$$3 \div \frac{1}{6} = 3 \times 6 = 18.$$

Students use story problems to make sense of division problems:[5.NF.7c]

> How much chocolate will each person get if 3 people share $\frac{1}{2}$ lb of chocolate equally? How many $\frac{1}{3}$-cup servings are in 2 cups of raisins?

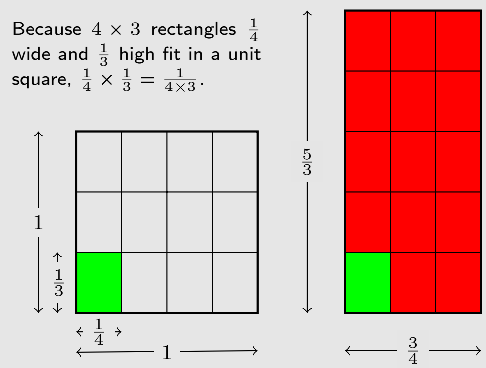
Students attend carefully to the underlying unit quantities when solving problems. For example, if $\frac{1}{2}$ of a fund-raiser's funds were raised by the 6th grade, and if $\frac{1}{3}$ of the 6th grade's funds were raised by Ms. Wilkin's class, then $\frac{1}{3} \times \frac{1}{2}$ gives the fraction of the fund-raiser's funds that Ms. Wilkin's class raised, but it does not tell us how much money Ms. Wilkin's class raised.[5.NF.6]

Multiplication as scaling In preparation for Grade 6 work in ratios and proportional reasoning, students learn to see products such as 5×3 or $\frac{1}{2} \times 3$ as expressions that can be interpreted in terms of a quantity, 3, and a scaling factor, 5 or $\frac{1}{2}$. Thus, in addition to knowing that $5 \times 3 = 15$, they can also say that 5×3 is 5 times as big as 3, without evaluating the product. Likewise, they see $\frac{1}{2} \times 3$ as half the size of 3.[5.NF.5a]

5.NF.4b Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

b Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

Using an area model to show that $\frac{3}{4} \times \frac{5}{3} = \frac{3 \times 5}{4 \times 3}$



The rectangle of width $\frac{3}{4}$ and height $\frac{5}{3}$ is tiled with 3×5 rectangles of area $\frac{1}{4 \times 3}$, so has area $\frac{3 \times 5}{4 \times 3}$.

5.NF.7abc Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.[2]

a Interpret division of a unit fraction by a non-zero whole number, and compute such quotients.

b Interpret division of a whole number by a unit fraction, and compute such quotients.

c Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem.

5.NF.6 Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

5.NF.5a Interpret multiplication as scaling (resizing), by:

a Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

The understanding of multiplication as scaling is an important opportunity for students to reasonin abstractly (MP2). Previous work with multiplication by whole numbers enables students to see multiplication by numbers bigger than 1 as producing a larger quantity, as when a recipe is doubled, for example. Grade 5 work with multiplying by unit fractions, and interpreting fractions in terms of division, enables students to see that multiplying a quantity by a number smaller than 1 produces a smaller quantity, as when the budget of a large state university is multiplied by $\frac{1}{2}$, for example.[5.NF.5b]

The special case of multiplying by 1, which leaves a quantity unchanged, can be related to fraction equivalence by expressing 1 as $\frac{n}{n}$, as explained on page 6.

5.NF.5b Interpret multiplication as scaling (resizing), by:

b Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

Progressions for the Common Core State Standards in Mathematics (draft)

26 December 2011

6–7, Ratios and Proportional Relationships

Overview

The study of ratios and proportional relationships extends students' work in measurement and in multiplication and division in the elementary grades. Ratios and proportional relationships are foundational for further study in mathematics and science and useful in everyday life. Students use ratios in geometry and in algebra when they study similar figures and slopes of lines, and later when they study sine, cosine, tangent, and other trigonometric ratios in high school. Students use ratios when they work with situations involving constant rates of change, and later in calculus when they work with average and instantaneous rates of change of functions. An understanding of ratio is essential in the sciences to make sense of quantities that involve derived attributes such as speed, acceleration, density, surface tension, electric or magnetic field strength, and to understand percentages and ratios used in describing chemical solutions. Ratios and percentages are also useful in many situations in daily life, such as in cooking and in calculating tips, miles per gallon, taxes, and discounts. They also are also involved in a variety of descriptive statistics, including demographic, economic, medical, meteorological, and agricultural statistics (e.g., birth rate, per capita income, body mass index, rain fall, and crop yield) and underlie a variety of measures, for example, in finance (exchange rate), medicine (dose for a given body weight), and technology (kilobits per second).

Ratios, rates, proportional relationships, and percent Ratios arise in situations in which two (or more) quantities are related.• Sometimes the quantities have the same units (e.g., 3 cups of apple juice and 2 cups of grape juice), other times they do not (e.g., 3 meters and 2 seconds). Some authors distinguish ratios from rates, using the term "ratio" when units are the same and "rate" when units are different; others use ratio to encompass both kinds of situations. The

• In the Standards, a quantity involves measurement of an attribute. Quantities may be discrete, e.g., 4 apples, or continuous, e.g., 4 inches. They may be measurements of physical attributes such as length, area, volume, weight, or other measurable attributes such as income. Quantities can vary with respect to another quantity. For example, the quantities "distance between the earth and the sun in miles," "distance (in meters) that Sharoya walked," or "my height in feet" vary with time.

Standards use ratio in the second sense, applying it to situations in which units are the same as well as to situations in which units are different. Relationships of two quantities in such situations may be described in terms of ratios, rates, percents, or proportional relationships.

A ratio associates two or more quantities. Ratios can be indicated in words as "3 to 2" and "3 for every 2" and "3 out of every 5" and "3 parts to 2 parts." This use might include units, e.g., "3 cups of flour for every 2 eggs" or "3 meters in 2 seconds." Notation for ratios can include the use of a colon, as in $3 : 2$. The quotient $\frac{3}{2}$ is sometimes called the value of the ratio $3 : 2$.•

• In everyday language. the word "ratio" sometimes refers to the value of a ratio, for example in the phrases "take the ratio of price to earnings" or "the ratio of circumference to diameter is π."

Ratios have associated rates. For example, the ratio 3 feet for every 2 seconds has the associated rate $\frac{3}{2}$ feet for every 1 second; the ratio 3 cups apple juice for every 2 cups grape juice has the associated rate $\frac{3}{2}$ cups apple juice for every 1 cup grape juice. In Grades 6 and 7, students describe rates in terms such as "for each 1," "for each," and "per." The *unit rate* is the numerical part of the rate; the "unit" in "unit rate" is often used to highlight the 1 in "for each 1" or "for every 1."

Equivalent ratios arise by multiplying each measurement in a ratio pair by the same positive number. For example, the pairs of numbers of meters and seconds in the margin are in equivalent ratios. Such pairs are also said to be in the same ratio. Proportional relationships involve collections of pairs of measurements in equivalent ratios. In contrast, a proportion is an equation stating that two ratios are equivalent. Equivalent ratios have the same unit rate.

Representing pairs in a proportional relationship

Sharoya walks 3 meters every 2 seconds. Let d be the number of meters Sharoya has walked after t seconds. d and t are in a proportional relationship.

d meters	3	6	9	12	15	$\frac{3}{2}$	1	2	4
t seconds	2	4	6	8	10	1	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{8}{3}$

d and t are related by the equation $d = \left(\frac{3}{2}\right) t$. Students sometimes use the equals sign incorrectly to indicate proportional relationships, for example, they might write "3 m = 2 sec" to represent the correspondence between 3 meters and 2 seconds. In fact, 3 meters is not equal to 2 seconds. This relationship can be represented in a table or by writing "3 m → 2 sec." Note that the unit rate appears in the pair $\left(\frac{3}{2}, 1\right)$.

The pairs of meters and seconds in the margin show distance and elapsed time varying together in a *proportional relationship*. This situation can be described as "distance traveled and time elapsed are proportionally related," or "distance and time are directly proportional," or simply "distance and time are proportional." The proportional relationship can be represented with the equation $d = \left(\frac{3}{2}\right) t$. The factor $\frac{3}{2}$ is the constant unit rate associated with the different pairs of measurements in the proportional relationship; it is known as a *constant of proportionality*.

The word *percent* means "per 100" (*cent* is an abbreviation of the Latin *centum* "hundred"). If 35 milliliters out of every 100 milliliters in a juice mixture are orange juice, then the juice mixture is 35% orange juice (by volume). If a juice mixture is viewed as made of 100 equal parts, of which 35 are orange juice, then the juice mixture is 35% orange juice.

More precise definitions of the terms presented here and a framework for organizing and relating the concepts are presented in the Appendix.

Recognizing and describing ratios, rates, and proportional relationships "For each," "for every," "per," and similar terms distinguish situations in which two quantities have a proportional relationship from other types of situations. For example, without further information "2 pounds for a dollar" is ambiguous. It may be that pounds and dollars are proportionally related and *every* two pounds costs a dollar. Or it may be that there is a discount on bulk, so weight and cost do not have a proportional relationship. Thus, recognizing ratios, rates, and proportional relationships involves looking for structure (MP7). Describing and interpreting descriptions of ratios, rates, and proportional relationships involves precise use of language (MP6).

Representing ratios, collections of equivalent ratios, rates, and proportional relationships Because ratios and rates are different and rates will often be written using fraction notation in high school, ratio notation should be distinct from fraction notation.

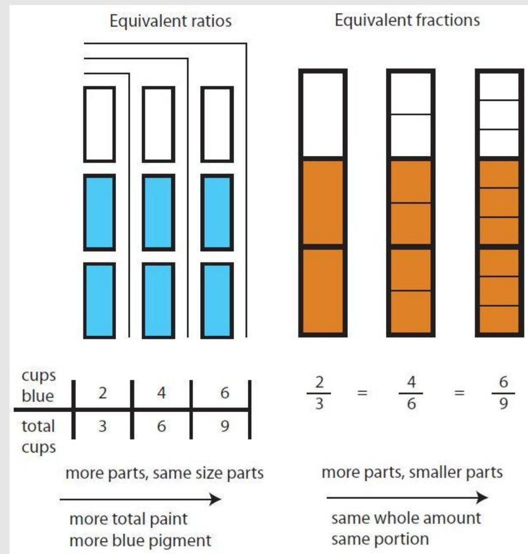
Together with tables, students can also use tape diagrams and double number line diagrams to represent collections of equivalent ratios. Both types of diagrams visually depict the relative sizes of the quantities.

Tape diagrams are best used when the two quantities have the same units. They can be used to solve problems and also to highlight the multiplicative relationship between the quantities.

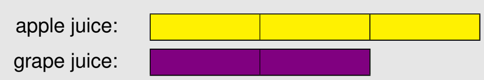
Double number line diagrams are best used when the quantities have different units (otherwise the two diagrams will use different length units to represent the same amount). Double number line diagrams can help make visible that there are many, even infinitely many, pairs in the same ratio, including those with rational number entries. As in tables, unit rates appear paired with 1.

A collection of equivalent ratios can be graphed in the coordinate plane. The graph represents a proportional relationship. The unit rate appears in the equation and graph as the slope of the line, and in the coordinate pair with first coordinate 1.

Equivalent ratios versus equivalent fractions

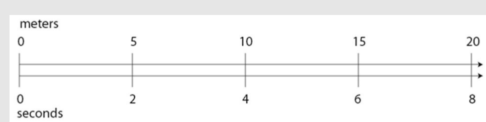


Representing ratios with tape diagrams



This diagram can be interpreted as representing any *mixture of apple juice and grape juice with a ratio of 3 to 2. The total amount of juice is represented as partitioned into 5 parts of equal size, represented by 5 rectangles. For example, if the diagram represents 5 cups of juice mixture, then each of these rectangles represents 1 cup. If the total amount of juice mixture is 1 gallon, then each part represents $\frac{1}{5}$ gallon and there are $\frac{3}{5}$ gallon of apple juice and $\frac{2}{5}$ gallon of grape juice.*

Representing ratios with double number line diagrams



On double number line diagrams, if A and B are in the same ratio, then A and B are located at the same distance from 0 on their respective lines. Multiplying A and B by a positive number p results in a pair of numbers whose distance from 0 is p times as far. So, for example, 3 times the pair 2 and 5 results in the pair 6 and 15 which is located at 3 times the distance from 0.

Grade 6

Representing and reasoning about ratios and collections of equivalent ratios Because the multiplication table is familiar to sixth graders, situations that give rise to columns or rows of a multiplication table can provide good initial contexts when ratios and proportional relationships are introduced. Pairs of quantities in equivalent ratios arising from whole number measurements such as "3 lemons for every $1" or "for every 5 cups grape juice, mix in 2 cups peach juice" lend themselves to being recorded in a table.[6.RP.3a] Initially, when students make tables of quantities in equivalent ratios, they may focus only on iterating the related quantities by repeated addition to generate equivalent ratios.

As students work with tables of quantities in equivalent ratios (also called ratio tables), they should practice using and understanding ratio and rate language.[6.RP.1,6.RP.2] It is important for students to focus on the meaning of the terms "for every," "for each," "for each 1," and "per" because these equivalent ways of stating ratios and rates are at the heart of understanding the structure in these tables, providing a foundation for learning about proportional relationships in Grade 7.

Students graph the pairs of values displayed in ratio tables on coordinate axes. The graph of such a collection of equivalent ratios lies on a line through the origin, and the pattern of increases in the table can be seen in the graph as coordinated horizontal and vertical increases.[6.EE.9]

6.RP.3a Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

a Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.

6.RP.1 Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities.

6.RP.2 Understand the concept of a unit rate a/b associated with a ratio $a : b$ with $b \neq 0$, and use rate language in the context of a ratio relationship.

6.EE.9 Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation.

Showing structure in tables and graphs

Additive Structure

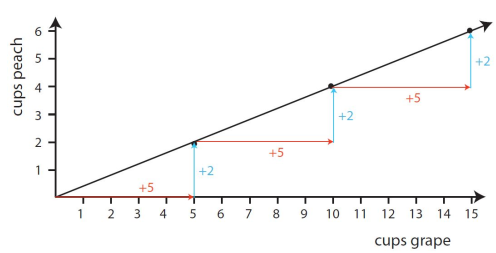
cups grape	cups peach
5	2
10	4
15	6
20	8
25	10

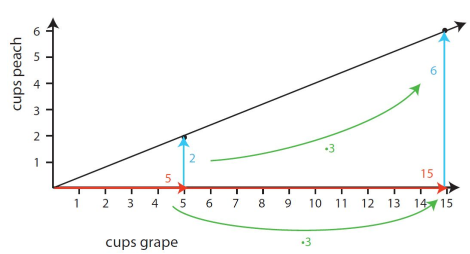
(+5 between successive cups grape entries; +2 between successive cups peach entries)

Multiplicative Structure

cups grape	cups peach
5	2
10	4
15	6
20	8
100	40

(·2, ·3, ·20 arrows from the first row on both sides)



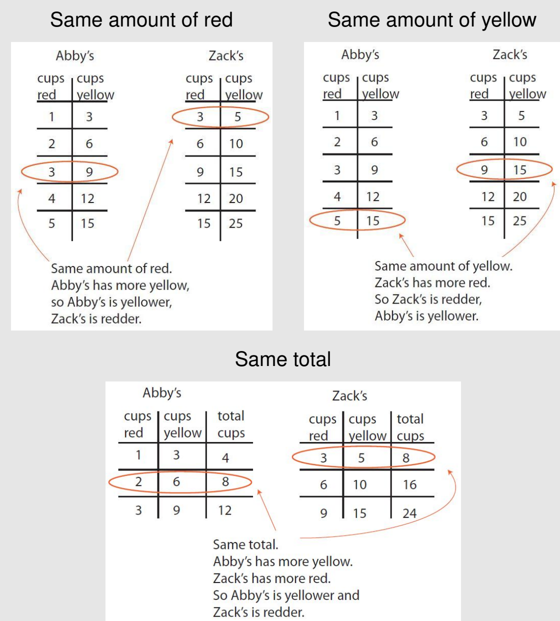


In the tables, equivalent ratios are generated by repeated addition (left) and by scalar multiplication (right). Students might be asked to identify and explain correspondences between each table and the graph beneath it (MP1).

By reasoning about ratio tables to compare ratios,[6.RP.3a] students can deepen their understanding of what a ratio describes in a context and what quantities in equivalent ratios have in common. For example, suppose Abby's orange paint is made by mixing 1 cup red paint for every 3 cups yellow paint and Zack's orange paint is made by mixing 3 cups red for every 5 cups yellow. Students could discuss that all the mixtures within a single ratio table for one of the paint mixtures are the same shade of orange because they are multiple batches of the same mixture. For example, 2 cups red and 6 cups yellow is two batches of 1 cup red and 3 cups yellow; each batch is the same color, so when the two batches are combined, the shade of orange doesn't change. Therefore, to compare the colors of the two paint mixtures, any entry within a ratio table for one mixture can be compared with any entry from the ratio table for the other mixture.

It is important for students to focus on the rows (or columns) of a ratio table as multiples of each other. If this is not done, a common error when working with ratios is to make additive comparisons. For example, students may think incorrectly that the ratios $1 : 3$ and $3 : 5$ of red to yellow in Abby's and Zack's paints are equivalent because the difference between the number of cups of red and yellow in both paints is the same, or because Zack's paint could be made from Abby's by adding 2 cups red and 2 cups yellow. The margin shows several ways students could reason correctly to compare the paint mixtures.

Three ways to compare paint mixtures

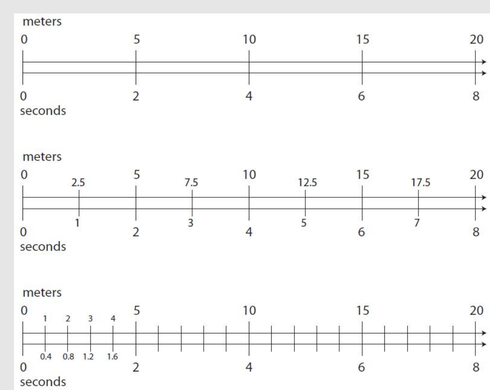


Strategies for solving problems Although it is traditional to move students quickly to solving proportions by setting up an equation, the Standards do not require this method in Grade 6. There are a number of strategies for solving problems that involve ratios. As students become familiar with relationships among equivalent ratios, their strategies become increasingly abbreviated and efficient.

For example, suppose grape juice and peach juice are mixed in a ratio of 5 to 2 and we want to know how many cups of grape juice to mix with 12 cups of peach juice so that the mixture will still be in the same ratio. Students could make a ratio table as shown in the margin, and they could use the table to find the grape juice entry that pairs with 12 cups of peach juice in the table. This perspective allows students to begin to reason about proportions by starting with their knowledge about multiplication tables and by building on this knowledge.

As students generate equivalent ratios and record them in tables, their attention should be drawn to the important role of multiplication and division in how entries are related to each other. Initially, students may fill ratio tables with columns or rows of the multiplication table by skip counting, using only whole number entries, and placing these entries in numerical order. Gradually, students should consider entries in ratio tables beyond those they find by skip counting, including larger entries and fraction or decimal entries. Finding

Double number line diagrams used for situations with different units



Double number lines indicate coordinated multiplying and dividing of quantities. This can also be indicated in tables.

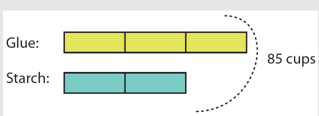
these other entries will require the explicit use of multiplication and division, not just repeated addition or skip counting. For example, if Seth runs 5 meters every 2 seconds, then Seth will run 2.5 meters in 1 second because in half the time he will go half as far. In other words, when the elapsed time is divided by 2, the distance traveled should also be divided by 2. More generally, if the elapsed time is multiplied (or divided) by N, the distance traveled should also be multiplied (or divided) by N. Double number lines can be useful in representing ratios that involve fractions and decimals.

As students become comfortable with fractional and decimal entries in tables of quantities in equivalent ratios, they should learn to appreciate that unit rates are especially useful for finding entries. A unit rate gives the number of units of one quantity per 1 unit of the other quantity. The amount for N units of the other quantity is then found by multiplying by N. Once students feel comfortable doing so, they may wish to work with abbreviated tables instead of working with long tables that have many values. The most abbreviated tables consist of only two columns or two rows; solving a proportion is a matter of finding one unknown entry in the table.

Measurement conversion provides other opportunities for students to use relationships given by unit rates.[6.RP.3d] For example, recognizing "12 inches in a foot," "1000 grams in a kilogram," or "one kilometer is $\frac{5}{8}$ of a mile" as rates, can help to connect concepts and methods developed for other contexts with measurement conversion.

Representing a problem with a tape diagram

Slimy Gloopy mixture is made by mixing glue and liquid laundry starch in a ratio of 3 to 2. How much glue and how much starch is needed to make 85 cups of Slimy Gloopy mixture?



5 parts $\longrightarrow$ 85 cups
1 part $\longrightarrow$ $85 \div 5 = 17$ cups
3 parts $\longrightarrow$ $3 \cdot 17 = 51$ cups
2 parts $\longrightarrow$ $2 \cdot 17 = 34$ cups

51 cups glue and 34 cups starch are needed.

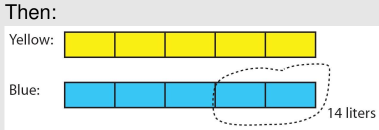
Tape diagrams can be useful aids for solving problems.

Representing a multi-step problem with two pairs of tape diagrams

Yellow and blue paint were mixed in a ratio of 5 to 3 to make green paint. After 14 liters of blue paint were added, the amount of yellow and blue paint in the mixture was equal. How much green paint was in the mixture at first?

At first:

Then:



2 parts $\longrightarrow$ 14 liters
1 part $\longrightarrow$ $14 \div 2 = 7$ liters
(original total) 8 parts $\longrightarrow$ $8 \cdot 7 = 56$ liters

There was 56 liters of green paint to start with.

This problem can be very challenging for sixth or seventh graders.

A progression of strategies for solving a proportion

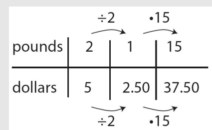
If 2 pounds of beans cost \$5, how much will 15 pounds of beans cost?

Method 1

pounds	2	4	6	8	10	12	14	1	15
dollars	5	10	15	20	25	30	35	2.50	37.50

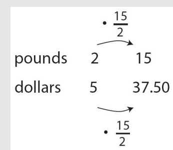
"I found 14 pounds costs \$35 and then 1 more pound is another \$2.50, so that makes \$37.50 in all."

Method 2



"I found 1 pound first because if I know how much it costs for each pound then I can find any number of pounds by multiplying."

Method 3



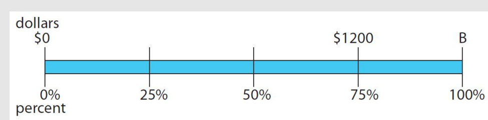
The previous method, done in one step.

With this perspective, the second column is seen as the first column times a number. To solve the proportion one first finds this number.

6.RP.3d Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.

Solving a percent problem

If 75% of the budget is \$1200, what is the full budget?



"I said 75% is 3 parts and is \$1200
25% is 1 part and is $\$1200 \div 3 = \$400$
100% is 4 parts and is $4 \cdot \$400 = \$1600$"

portion	75	3	1200
whole	100	4	1600

75% is $\frac{1200}{B}$ $\qquad$ 75% of B is 1200

$\frac{75}{100} = \frac{1200}{B}$ $\qquad$ $\frac{75}{100} \cdot B = 1200$

$B = 1600$

In reasoning about and solving percent problems, students can use a variety of strategies. Representations such as this, which is a blend between a tape diagram and a double number line diagram, can support sense-making and reasoning about percent.

Grade 7

In Grade 7, students extend their reasoning about ratios and proportional relationships in several ways. Students use ratios in cases that involve pairs of rational number entries, and they compute associated unit rates. They identify these unit rates in representations of proportional relationships. They work with equations in two variables to represent and analyze proportional relationships. They also solve multi-step ratio and percent problems, such as problems involving percent increase and decrease.

At this grade, students will also work with ratios specified by rational numbers, such as $\frac{3}{4}$ cups flour for every $\frac{1}{2}$ stick butter.[7.RP.1] Students continue to use ratio tables, extending this use to finding unit rates.

Recognizing proportional relationships Students examine situations carefully, to determine if they describe a proportional relationship.[7.RP.2a] For example, if Josh is 10 and Reina is 7, how old will Reina be when Josh is 20? We cannot solve this problem with the proportion $\frac{10}{7} = \frac{20}{R}$ because it is not the case that for every 10 years that Josh ages, Reina ages 7 years. Instead, when Josh has aged 10 another years, Reina will as well, and so she will be 17 when Josh is 20.

For example, if it takes 2 people 5 hours to paint a fence, how long will it take 4 people to paint a fence of the same size (assuming all the people work at the same steady rate)? We cannot solve this problem with the proportion $\frac{2}{5} = \frac{4}{H}$ because it is not the case that for every 2 people, 5 hours of work are needed to paint the fence. When more people work, it will take fewer hours. With twice as many people working, it will take half as long, so it will take only 2.5 hours for 4 people to paint a fence. Students must understand the structure of the problem, which includes looking for and understand the roles of "for every," "for each," and "per."

Students recognize that graphs that are not lines through the origin and tables in which there is not a constant ratio in the entries do not represent proportional relationships. For example, consider circular patios that could be made with a range of diameters. For such patios, the area (and therefore the number of pavers it takes to make the patio) is not proportionally related to the diameter, although the circumference (and therefore the length of stone border it takes to encircle the patio) is proportionally related to the diameter. Note that in the case of the circumference, C, of a circle of diameter D, the constant of proportionality in $C = \pi \cdot D$ is the number π, which is not a rational number.

Equations for proportional relationships As students work with proportional relationships, they write equations of the form $y = cx$, where c is a constant of proportionality, i.e., a unit rate.[7.RP.2c] They

7.RP.1 Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units.

Ratio problem specified by rational numbers: Three approaches

To make Perfect Purple paint mix $\frac{1}{2}$ cup blue paint with $\frac{1}{3}$ cup red paint. If you want to mix blue and red paint in the same ratio to make 20 cups of Perfect Purple paint, how many cups of blue paint and how many cups of red paint will you need?

Method 1

cups blue	$\frac{1}{2}$	3	12
cups red	$\frac{1}{3}$	2	8
total cups purple	$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$	5	20

(Arrows: $\cdot 6$ from first to second column, $\cdot 4$ from second to third column.)

"I thought about making 6 batches of purple because that is a whole number of cups of purple. To make 6 batches, I need 6 times as much blue and 6 times as much red too. That was 3 cups blue and 2 cups red and that made 5 cups purple. Then 4 times as much of each makes 20 cups purple."

Method 2

$$\frac{1}{2} \div \frac{5}{6} = \frac{1}{2} \cdot \frac{6}{5} = \frac{6}{10} \qquad \frac{6}{10} \cdot 20 = 12$$

$$\frac{1}{3} \div \frac{5}{6} = \frac{1}{3} \cdot \frac{6}{5} = \frac{6}{15} \qquad \frac{6}{15} \cdot 20 = 8$$

"I found out what fraction of the paint is blue and what fraction is red. Then I found those fractions of 20 to find the number of cups of blue and red in 20 cups."

Method 3

cups blue	$\frac{1}{2}$	12
cups red	$\frac{1}{3}$	8
total cups purple	$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$	20

(Arrows in each column: $\cdot \frac{2}{5}$ from total to red, $\cdot \frac{3}{5}$ from total to blue.)

Like Method 2, but in tabular form, and viewed as multiplicative comparisons.

7.RP.2a Recognize and represent proportional relationships between quantities.

a Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.

7.RP.2c Represent proportional relationships by equations.

see this unit rate as the amount of increase in y as x increases by 1 unit in a ratio table and they recognize the unit rate as the vertical increase in a "unit rate triangle" or "slope triangle" with horizontal side of length 1 for a graph of a proportional relationship.[7.RP.2b]

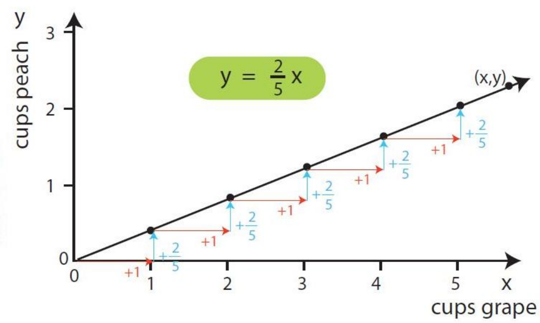
7.RP.2b Recognize and represent proportional relationships between quantities.

b Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.

Correspondence among a table, graph, and equation of a proportional relationship

For every 5 cups grape juice, mix in 2 cups peach juice.

x cups grape	y cups peach
(0)	(0)
5	2
1	$\frac{2}{5}$
2	$2 \cdot \frac{2}{5}$
3	$3 \cdot \frac{2}{5}$
4	$4 \cdot \frac{2}{5}$
x	$x \cdot \frac{2}{5}$



On the graph: For each 1 unit you move to the right, move up $\frac{2}{5}$ of a unit.

When you go 2 units to the right, you go up $2 \cdot \frac{2}{5}$ units.

When you go 3 units to the right, you go up $3 \cdot \frac{2}{5}$ units.

When you go 4 units to the right, you go up $4 \cdot \frac{2}{5}$ units.

When you go x units to the right, you go up $x \cdot \frac{2}{5}$ units.

Starting from $(0, 0)$, to get to a point (x, y) on the graph, go x units to the right, so go up $x \cdot \frac{2}{5}$ units.

Therefore $y = x \cdot \frac{2}{5}$ $y = \frac{2}{5}x$

Students connect their work with equations to their work with tables and diagrams. For example, if Seth runs 5 meters every 2 seconds, then how long will it take Seth to run 100 meters at that rate? The traditional method is to formulate an equation, $\frac{5}{2} = \frac{100}{T}$, cross-multiply, and solve the resulting equation to solve the problem. If $\frac{5}{2}$ and $\frac{100}{T}$ are viewed as unit rates obtained from the equivalent ratios $5 : 2$ and $100 : T$, then they must be equivalent fractions because equivalent ratios have the same unit rate. To see the rationale for cross-multiplying, note that when the fractions are given the common denominator $2 \cdot T$, then the numerators become $5 \cdot T$ and $2 \cdot 100$ respectively. Once the denominators are equal, the fractions are equal exactly when their numerators are equal, so $5 \cdot T$ must equal $2 \cdot 100$ for the unit rates to be equal. This is why we can solve the equation $5 \cdot T = 2 \cdot 100$ to find the amount of time it will take for Seth to run 100 meters.

A common error in setting up proportions is placing numbers in incorrect locations. This is especially easy to do when the order in which quantities are stated in the problem is switched within the problem statement. For example, the second of the following two problem statements is more difficult than the first because of the reversal.

> "If a factory produces 5 cans of dog food for every 3 cans of cat food, then when the company produces 600 cans of dog food, how many cans of cat food will it produce?"
>
> "If a factory produces 5 cans of dog food for every 3 cans of cat food, then how many cans of cat food will the company produce when it produces 600 cans of dog food?"

Such problems can be framed in terms of proportional relationships and the constant of proportionality or unit rate, which is obscured by the traditional method of setting up proportions. For example, if Seth runs 5 meters every 2 seconds, he runs at a rate of 2.5 meters per second, so distance d (in meters) and time t (in seconds) are related by $d = 2.5t$. If $d = 100$ then $t = \frac{100}{2.5} = 40$, so he takes 40 seconds to run 100 meters.

Multistep problems Students extend their work to solving multistep ratio and percent problems.[7.RP.3] Problems involving percent increase or percent decrease require careful attention to the referent whole. For example, consider the difference in these two percent decrease and percent increase problems:

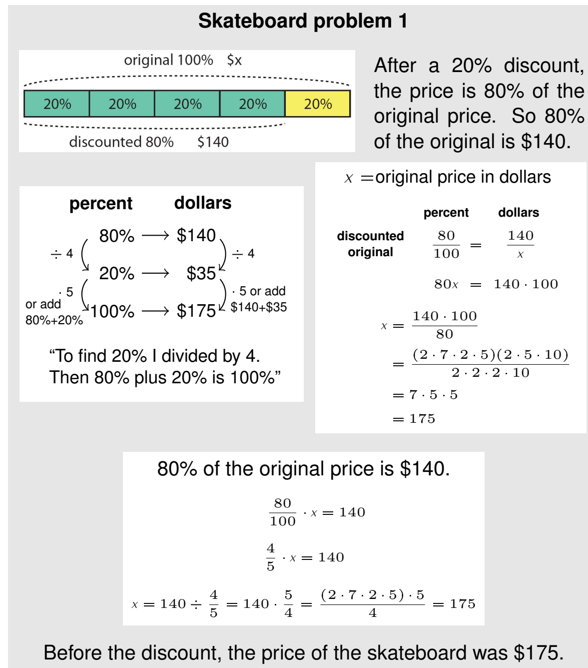
> *Skateboard problem 1.* After a 20% discount, the price of a SuperSick skateboard is \$140. What was the price before the discount?
>
> *Skateboard problem 2.* A SuperSick skateboard costs \$140 now, but its price will go up by 20%. What will the new price be after the increase?

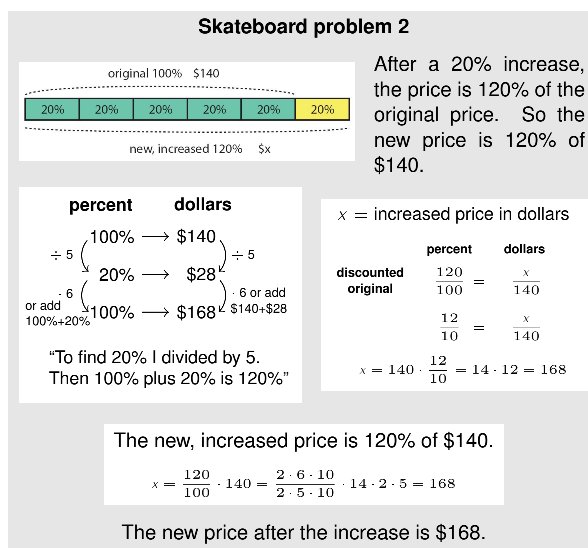
The solutions to these two problems are different because the 20% refers to different wholes or 100% amounts. In the first problem, the 20% is 20% of the larger pre-discount amount, whereas in the second problem, the 20% is 20% of the smaller pre-increase amount. Notice that the distributive property is implicitly involved in working with percent decrease and increase. For example, in the first problem, if x is the original price of the skateboard (in dollars), then after the 20% discount, the new price is $x - 20\% \cdot x$. The distributive property shows that the new price is $80\% \cdot x$:

$$x - 20\% \cdot x = 100\% \cdot x - 20\% \cdot x = (100\% - 20\%)x = 80\% \cdot x$$

Percentages can also be used in making comparisons between two quantities. Students must attend closely to the wording of such problems to determine what the whole or 100% amount a percentage refers to.

7.RP.3 Use proportional relationships to solve multistep ratio and percent problems.





Connection to Geometry One new context for proportions at Grade 7 is scale drawings.[7.G.1] To compute unknown lengths from known lengths, students can set up proportions in tables or equations, or they can reason about how lengths compare multiplicatively. Students can use two kinds of multiplicative comparisons. They can apply a scale factor that relates lengths in two different figures, or they can consider the ratio of two lengths within one figure, find a multiplicative relationship between those lengths, and apply that relationship to the ratio of the corresponding lengths in the other figure. When working with areas, students should be aware that areas do not scale by the same factor that relates lengths. (Areas scale by the square of the scale factor that relates lengths, if area is measured in the unit of measurement derived from that used for length.)

Connection to Statistics and Probability Another new context for proportions at Grade 7 is to drawing inferences about a population from a random sample.[7.SP.1] Because random samples can be expected to be approximately representative of the full population, one can imagine selecting many samples of that same size until the full population is exhausted, each with approximately the same characteristics. Therefore the ratio of the size of a portion having a certain characteristic to the size of the whole should be approximately the same for the sample as for the full population.

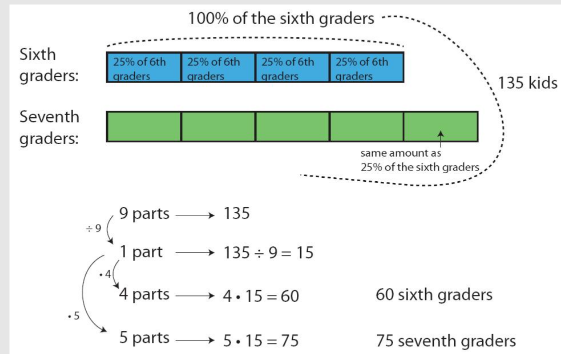
Where the Ratios and Proportional Relationships Progression is heading

The study of proportional relationships is a foundation for the study of functions, which continues through High School and beyond. Linear functions are characterized by having a constant rate of change (the change in the outputs is a constant multiple of the change in the corresponding inputs). Proportional relationships are a major type of linear function; they are those linear functions that have a positive rate of change and take 0 to 0.

Students extend their understanding of quantity. They write rates concisely in terms of derived units such as mi/hr rather than expressing them in terms such as "$\frac{3}{2}$ miles in every 1 hour." They encounter a wider variety of derived units and situations in which they must conceive units that measure attributes of interest.

Using percentages in comparisons

There are 25% more seventh graders than sixth graders in the after-school club. If there are 135 sixth and seventh graders altogether in the after-school club, how many are sixth graders and how many are seventh graders?



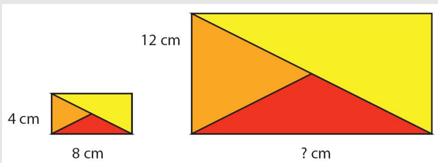
"25% more seventh graders than sixth graders means that the number of extra seventh graders is the same as 25% of the sixth graders."

[7.G.1] Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

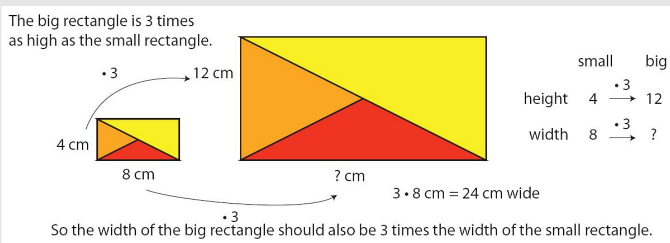
[7.SP.1] Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.

Connection to geometry

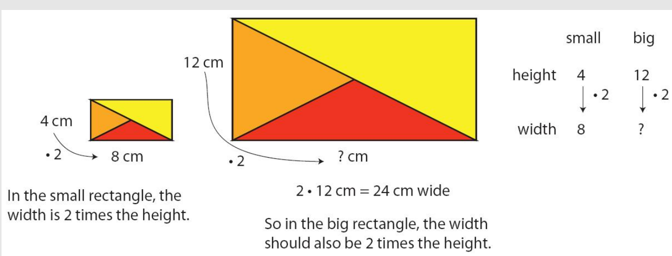
If the two rectangles are similar, then how wide is the larger rectangle?



Use a scale factor: Find the scale factor from the small rectangle to the larger one:



Use an internal comparison: Compare the width to the height in the small rectangle. The ratio of the width to height is the same in the large rectangle.



Connection to statistics and probability

There are 150 tiles in a bin. Some of the tiles are blue and the rest are yellow. A random sample of 10 tiles was selected. Of the 10 tiles, 3 were yellow and 7 were blue. What are the best estimates for how many blue tiles are in the bin and how many yellow tiles are in the bin?

Student 1

yellow:	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
blue:	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105
total:	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150

"I figured if you keep picking out samples of 10 they should all be about the same, so I got this ratio table. Out of 150 tiles, about 45 should be yellow and about 105 should be blue."

Student 2

yellow:	3	45
blue:	7	105
total:	10	150

• 15

"I also made a ratio table. I said that if there are 15 times as many tiles in the bin as in the sample, then there should be about 15 times as many yellow tiles and 15 times as many blue tiles. $15 \cdot 3 = 45$, so 45 yellow tiles. $15 \cdot 7 = 105$, so 105 blue tiles."

Student 3

30% yellow tiles $\quad 30\% \cdot 150 = \frac{3 \cdot 10}{10 \cdot 10} \cdot 150 = \frac{3}{10} \cdot 15 \cdot 10 = 45$

70% blue tiles $\quad 70\% \cdot 150 = \frac{7 \cdot 10}{10 \cdot 10} \cdot 150 = \frac{7}{10} \cdot 15 \cdot 10 = 105$

"I used percentages. 3 out of 10 is 30% yellow and 7 out of 10 is 70% blue. The percentages in the whole bin should be about the same as the percentages in the sample."

Appendix: A framework for ratio, rate, and proportional relationships

This section presents definitions of the terms ratio, rate, and proportional relationship that are consistent with the Standards and it briefly summarizes some of the essential characteristics of these concepts. It also provides an organizing framework for these concepts. Because many different authors have used ratio and rate terminology in widely differing ways, there is a need to standardize the terminology for use with the Standards and to have a common framework for curriculum developers, professional development providers, and other education professionals to discuss the concepts. This section does not describe how the concepts should be presented to students in Grades 6 and 7.

Definitions and essential characteristics of ratios, rates, and proportional relationships

A *ratio* is a pair of non-negative numbers, $A : B$, which are not both 0.

When there are A units of one quantity for every B units of another quantity, a *rate* associated with the ratio $A : B$ is $\frac{A}{B}$ units of the first quantity per 1 unit of the second quantity. (Note that the two quantities may have different units.) The associated *unit rate* is $\frac{A}{B}$. The term *unit rate* is the numerical part of the rate; the "unit" is used to highlight the 1 in "per 1 unit of the second quantity." Unit rates should not be confused with unit fractions (which have a 1 in the numerator).

A rate is expressed in terms of a unit that is derived from the units of the two quantities (such as m/s, which is derived from meters and seconds). In high school and beyond, a rate is usually written as

$$\frac{A}{B}\frac{\text{units}}{\texttt{UNITS}}$$

where the two different fonts highlight the possibility that the quantities may have different units. In practice, when working with a ratio $A : B$, the rate $\frac{A}{B}$ units per 1 **UNIT** and the rate $\frac{B}{A}$ **UNITS** per 1 unit are both useful.

The *value* of a ratio $A : B$ is the quotient $\frac{A}{B}$ (if B is not 0). Note that the value of a ratio may be expressed as a decimal, percent, fraction, or mixed number. The value of the ratio $A : B$ tells how A and B compare multiplicatively; specifically, it tells how many times as big A is as B. In practice, when working with a ratio $A : B$, the value $\frac{A}{B}$ as well as the value $\frac{B}{A}$, associated with the ratio $B : A$, are both useful. These values of each ratio are viewed as unit rates in some contexts (see Perspective 1 in the next section).

Two ratios $A : B$ and $C : D$ are *equivalent* if there is a positive number, c, such that $C = cA$ and $D = cB$. To check that two ratios are equivalent one can check that they have the same value (because $\frac{cA}{cB} = \frac{A}{B}$), or one can "cross-multiply" and check that $A \cdot D = B \cdot C$ (because $A \cdot cB = B \cdot cA$). Equivalent ratios have the same unit rate.

A *proportional relationship* is a collection of pairs of numbers that are in equivalent ratios. A ratio $A : B$ determines a proportional relationship, namely the collection of pairs (cA, cB), for c positive. A proportional relationship is described by an equation of the form $y = kx$, where k is a positive constant, often called a *constant of proportionality*. The constant of proportionality, k, is equal to the value $\frac{B}{A}$. The graph of a proportional relationship lies on a ray with endpoint at the origin.

Two perspectives on ratios and their associated rates in quantitative contexts

Although ratios, rates, and proportional relationships can be described in purely numerical terms, these concepts are most often used with quantities.

Ratios are often described as comparisons by division, especially when focusing on an associated rate or value of the ratio. There are also two broad categories of basic ratio situations. Some division situations, notably those involving area, can fit into either category of division. Many ratio situations can be viewed profitably from within either category of ratio. For this reason, the two categories for ratio will be described as *perspectives* on ratio.

First perspective: Ratio as a composed unit or batch Two quantities are in a ratio of A to B if for every A units present of the first quantity there are B units present of the second quantity. In other words, two quantities are in a ratio of A to B if there is a positive number c (which could be a rational number), such that there are $c \cdot A$ units of the first quantity and $c \cdot B$ units of the second quantity. With this perspective, the two quantities can have the same or different units.

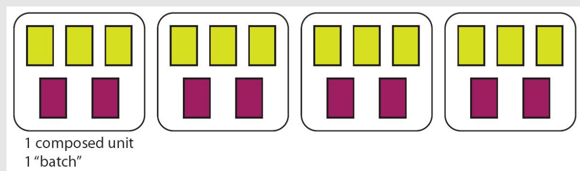
With this perspective, a ratio is specified by a composed unit or "batch," such as "3 feet in 2 seconds," and the unit or batch can be repeated or subdivided to create new pairs of amounts that are in the same ratio. For example, 12 feet in 8 seconds is in the ratio 3 to 2 because for every 3 feet, there are 2 seconds. Also, 12 feet in 8 seconds can be viewed as a 4 repetitions of the unit "3 feet in 2 seconds." Similarly, $\frac{3}{2}$ feet in 1 second is $\frac{1}{2}$ of the unit "3 feet in 2 seconds."

With this perspective, quantities that are in a ratio A to B give rise to a rate of $\frac{A}{B}$ units of the first quantity for every 1 unit of the second quantity (as well as to the rate of $\frac{B}{A}$ units of the second quantity for every 1 unit of the first quantity). For example, the ratio 3 feet in 2 seconds gives rise to the rate $\frac{3}{2}$ feet for every 1 second.

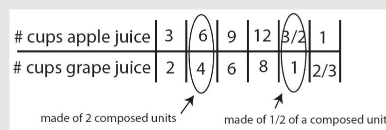
Two perspectives on ratio

1) There are 3 cups of apple juice for every 2 cups of grape juice in the mixture.

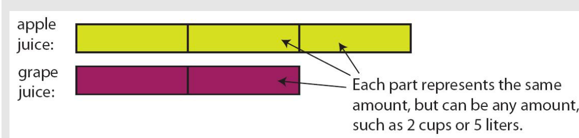
This way uses a composed unit: 3 cups apple juice and 2 cups grape juice. Any mixture that is made from some number of the composed unit is in the ratio 3 to 2.



In each of these mixtures, apple juice and grape juice are mixed in a ratio of 3 to 2:



2) The mixture is made from 3 parts apple juice and 2 parts grape juice, where all parts are the same size, but can be any size.



If 1 part is :	1 cup	2 cups	5 liters	3 quarts
amt of apple juice:	3 cups	6 cups	15 liters	9 quarts
amt of grape juice:	2 cups	4 cups	10 liters	6 quarts

With this perspective, if the relationship of the two quantities is represented by an equation $y = cx$, the constant of proportionality, c, can be viewed as the numerical part of a rate associated with the ratio $A : B$.

Second perspective: Ratio as fixed numbers of parts Two quantities which have the same units, are in a ratio of A to B if there is a part of some size such that there are A parts present of the first quantity and B parts present of the second quantity. In other words, two quantities are in a ratio of A to B if there is a positive number c (which could be a rational number), such that there are $A \cdot c$ units of the first quantity and $B \cdot c$ units of the second quantity.

With this perspective, one thinks of a ratio as two pieces. One piece is constituted of A parts, the other of B parts. To create pairs of measurements in the same ratio, one specifies an amount and fills each part with that amount. For example, in a ratio of 3 parts sand to 2 parts cement, each part could be filled with 5 cubic yards, so that there are 15 cubic yards of sand and 10 cubic yards of cement; or each part could be filled with 10 cubic meters, so that there are 30 cubic meters of sand and 20 cubic meters of cement. When describing a ratio from this perspective, the units need not be explicitly stated, as in "mix sand and cement in a ratio of 3 to 2." However, the type of quantity must be understood or stated explicitly, as in "by volume" or "by weight."

With this perspective, a ratio $A : B$ has an associated value, $\frac{A}{B}$, which describes how the two quantities are related multiplicatively. Specifically, $\frac{A}{B}$ is the factor that tells how many times as much of the first quantity there is as of the second quantity. (Similarly, the factor $\frac{B}{A}$ associated with the ratio $B : A$, tells how many times as much of the second quantity there is as of the first quantity.) For example, if sand and cement are mixed in a ratio of 3 to 2, then there is $\frac{3}{2}$ times as much sand as cement and there is $\frac{2}{3}$ times as much cement as sand.

With this second perspective, if the relationship of the two quantities is represented by an equation $y = cx$, the constant of proportionality, c, can be considered a factor that does not have a unit.

Progressions for the Common Core State Standards in Mathematics (draft)

3 December 2012

Grade 8, High School, Functions*

Overview

Functions describe situations in which one quantity is determined by another. The area of a circle, for example, is a function of its radius. When describing relationships between quantities, the defining characteristic of a *function* is that the input value determines the output value or, equivalently, that the output value depends upon the input value.

The mathematical meaning of function is quite different from some common uses of the word, as in, "One function of the liver is to remove toxins from the body," or "The party will be held in the function room at the community center." The mathematical meaning of function is close, however, to some uses in everyday language. For example, a teacher might say, "Your grade in this class is a function of the effort you put into it." A doctor might say, "Some illnesses are a function of stress." Or a meteorologist might say, "After a volcano eruption, the path of the ash plume is a function of wind and weather." In these examples, the meaning of "function" is close to its mathematical meaning.

In some situations where two quantities are related, each can be viewed as a function of the other. For example, in the context of rectangles of fixed perimeter, the length can be viewed as depending upon the width or vice versa. In some of these cases, a problem context may suggest which one quantity to choose as the input variable.

Undergraduate mathematics may involve functions of more than one variable. The area of a rectangle, for example, can be viewed as a function of two variables: its width and length. But in high school mathematics the study of functions focuses primarily on real-valued functions of a single real variable, which is to say that both the input and output values are real numbers. One exception is in high school geometry, where geometric transformations are considered to be functions.• For example, a translation T, which moves the plane

• G-CO.2 . . . [D]escribe transformations as functions that take points in the plane as inputs and give other points as outputs. . . .

*Advanced material, corresponding to (+) standards, is indicated by plus signs in the left margin.

3 units to the right and 2 units up might be represented by $T : (x, y) \mapsto (x + 3, y + 2)$.

A trickle of pattern standards in Grades 4 and 5 begins the preparation for functions.[4.OA.5, 5.OA.3] Note that in both these standards a rule is explicitly given. Traditional pattern activities, where students are asked to continue a pattern through observation, are not a mathematical topic, and do not appear in the Standards in their own right.[1]

4.OA.5 Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself.

5.OA.3 Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane.

The Grade 4–5 pattern standards expand to a full progression on Ratios and Proportional Relationships in Grades 6–7, and then the notion of a function is introduced in Grade 8.

Before they learn the term "function," students begin to gain experience with functions in elementary grades. In Kindergarten, they use patterns with numbers such as the $5 + n$ pattern to learn particular additions and subtractions.

Sequences and functions Patterns are sequences, and sequences are functions with a domain consisting of whole numbers. However, in many elementary patterning activities, the input values are not given explicitly. In high school, students learn to use an index to indicate which term is being discussed. Erica handles this issue in the example in the margin by deciding that the term 2 would correspond to an index value of 1. Then the terms 4, 6, and 8 would correspond to input values of 2, 3, and 4, respectively. Erica could have decided that the term 2 would correspond to a different index value, such as 0. The resulting formula would have been different, but the (unindexed) sequence would have been the same.

Functions and modeling In modeling situations, knowledge of the context and statistics are sometimes used together to find algebraic expressions that best fit an observed relationship between quantities. Then the algebraic expressions can be used to interpolate (i.e., approximate or predict function values between and among the collected data values) and to extrapolate (i.e., to approximate or predict function values beyond the collected data values). One must always ask whether such approximations are reasonable in the context.

In school mathematics, functional relationships are often given by algebraic expressions. For example, $f(n) = n^2$ for $n \geqslant 1$ gives the n^{th} square number. But in many modeling situations, such as the temperature at Boston's Logan Airport as a function of time, algebraic expressions are generally not suitable.

Functions and Algebra See the Algebra Progression for a discussion of the connection and distinctions between functions, on the one hand, and algebra and equation solving, on the other. Perhaps the

[1] This does not exclude activities where patterns are used to support other standards, as long as the case can be made that they do so.

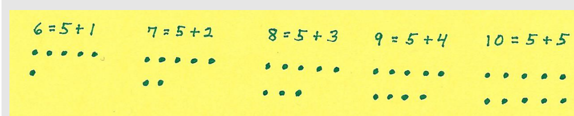
The problem with patterns

Students are asked to continue the pattern 2, 4, 6, 8, ... Here are some legitimate responses:

- Cody: I am thinking of a "plus 2 pattern," so it continues 10, 12, 14, 16, ...
- Ali: I am thinking of a repeating pattern, so it continues 2, 4, 6, 8, 2, 4, 6, 8, ...
- Suri: I am thinking of the units digit in the multiples of 2, so it continues 0, 2, 4, 6, 8, 0, 2, ...
- Erica: If $g(n)$ is any polynomial, then $f(n) = 2n + (n-1)(n-2)(n-3)(n-4)g(n)$ describes a continuation of this sequence.
- Zach: I am thinking of that high school cheer, "Who do we appreciate?"

Because the task provides no structure, all of these answers must be considered correct. Without any structure, continuing the pattern is simply speculation—a guessing game. Because there are infinitely many ways to continue a sequence, patterning problems should provide enough structure so that the sequence is well defined.

Experiences with functions before Grade 8



$f(n) = 5 + n$ *(pattern from Kindergarten, p. ??)*

$$1 \times 9 = 9$$
$$2 \times 9 = 2 \times (10 - 1) = (2 \times 10) - (2 \times 1) = 20 - 2 = 18$$
$$3 \times 9 = 3 \times (10 - 1) = (3 \times 10) - (3 \times 1) = 30 - 3 = 27,$$

$f(n) = 9 \times n = 10 \times n - n$ *(pattern from grade 3, p. ??)*

feet	inches
0	0
1	12
2	24
3	

$f(t) = 12t$ *(foot and inch equivalences from grade 4, p. ??)*

d meters	3	6	9	12	15	$\frac{3}{2}$	1	2	4
t seconds	2	4	6	8	10	1	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{8}{3}$

$f(t) = \frac{3}{2}t$ *(proportional relationship from Grade 6, p. ??)*

most productive connection is that solving equations can be seen as finding the intersections of graphs of functions.[A-REI.11]

A-REI.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

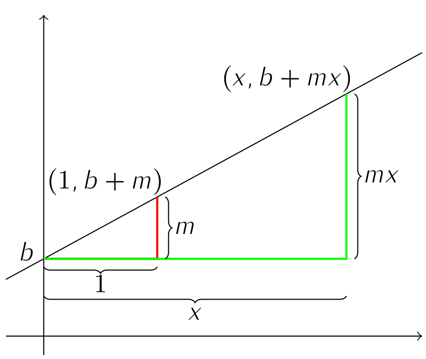
What to expect from this document The study of functions occupies a large part of a student's high school career, and this document does not treat in detail all of the material studied. Rather it gives some general guidance about ways to treat the material and ways to tie it together.

Grade 8

Define, evaluate, and compare functions Since the elementary grades, students have been describing patterns and expressing relationships between quantities. These ideas become semi-formal in Grade 8 with the introduction of the concept of *function*: a rule that assigns to each input exactly one output.[8.F.1] Formal language, such as domain and range, and function notation may be postponed until high school.

Building on experience with graphs and tables in Grades 6 and 7, students establish a routine of exploring functional relationships algebraically, graphically, numerically in tables, and through verbal descriptions.[8.F.2, MP1] And to develop flexibility in interpreting and translating among these various representations, students compare two functions represented in different ways, as illustrated by the task in the margin.

The main focus in Grade 8 is linear functions, those of the form $y = mx + b$, where m and b are constants.[8.F.3] The proof that $y = mx + b$ is also the equation of a line, and hence that the graph of a linear function is a line, is an important pieces of reasoning connecting algebra with geometry in Grade 8.[8.EE.6]



In the figure above, the red triangle is the "slope triangle" formed by the vertical intercept and the point on the line with x-coordinate equal to 1. The green triangle is formed from the intercept and a point with arbitrary x-coordinate. A dilation with center at the vertical intercept and scale factor x takes the red triangle to the green triangle, because it takes lines to parallel lines. Thus the green triangle is similar to the red triangle, and so the height of the green triangle is mx, and the coordinates of the general point on the triangle are $(x, b + mx)$. Which is to say that the point satisfies the equation $y = b + mx$.

Students learn to recognize linearity in a table: when constant differences between input values produce constant differences between output values. And they can use the constant rate of change appropriately in a verbal description of a context.

[8.F.1] Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.Function notation is not required in Grade 8.

[8.F.2] Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

[MP1] "Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends"

Battery charging

Sam wants to take his MP3 player and his video game player on a car trip. An hour before they plan to leave, he realized that he forgot to charge the batteries last night. At that point, he plugged in both devices so they can charge as long as possible before they leave.
Sam knows that his MP3 player has 40% of its battery life left and that the battery charges by an additional 12 percentage points every 15 minutes.
His video game player is new, so Sam doesn't know how fast it is charging but he recorded the battery charge for the first 30 minutes after he plugged it in.

time charging	0	10	20	30
video game player batter charge	20	32	44	56

1. If Sam's family leaves as planned, what percent of the battery will be charged for each of the two devices when they leave?
2. How much time would Sam need to charge the battery 100% on both devices?

For solutions and more discussion of this task, go to Illustrative Mathematics at illustrativemathematics.org/illustrations/641

[8.F.3] Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear.

[8.EE.6] Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

Use functions to model relationships between quantities When using functions to model a linear relationships between quantities, students learn to determine the rate of change of the function, which is the slope of its graph. They can read (or compute or approximate) the rate of change from a table or a graph, and they can interpret the rate of change in context.[8.F.4]

Graphs are ubiquitous in the study of functions, but it is important to distinguish a function from its graph. For example, a function does not have a slope but its graph can have a slope.•

Within the class of linear functions, students learn that some are proportional relationships and some are not. Functions of the form $y = mx + b$ are proportional relationships exactly when $b = 0$, so that y is proportional to x. Graphically, a linear function is a proportional relationship if its graph goes through the origin.

To understand relationships between quantities, it is often helpful to describe the relationships qualitatively, paying attention to the general shape of the graph without concern for specific numerical values.[8.F.5] The standard approach proceeds from left to right, describing what happens to the output as the input value increases. For example, pianist Chris Donnelly describes the relationship between creativity and structure via a graph.•

The qualitative description might be as follows: "As the input value (structure) increases, the output (creativity) increases quickly at first and gradually slowing down. As input (structure) continues to increase, the output (creativity) reaches a maximum and then starts decreasing, slowly at first, and gradually faster." Thus, from the graph alone, one can infer Donnelly's point that there is an optimal amount of structure that produces maximum creativity. With little structure or with too much structure, in contrast, creativity is low.

[8.F.4] Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

• The slope of a vertical line is undefined and the slope of a horizontal line is 0. Either of these cases might be considered "no slope." Thus, the phrase "no slope" should be avoided because it is imprecise and unclear.

[8.F.5] Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

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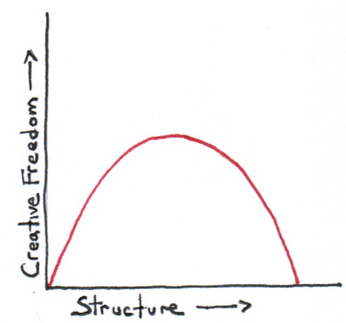


High School

The high school standards on functions are organized into four groups: Interpreting Functions (F-IF); Building Functions (F-BF); Linear, Quadratic and Exponential Models (F-LE); and Trigonometric Functions (F-TF). The organization of the first two groups under mathematical practices rather than types of function is an important aspect of the Standards: students should develop ways of thinking that are general and allow them to approach any type of function, work with it, and understand how it behaves, rather than see each function as a completely different animal in the bestiary. For example, they should see linear and exponential functions as arising out of structurally similar growth principles; they should see quadratic, polynomial, and rational functions as belonging to the same system (helped along by the unified study in the Algebra category of Arithmetic with Polynomials and Rationl Expressions).

Interpreting Functions

Understand the concept of a function and use function notation Building on semi-formal notions of functions from Grade 8, students in high school begin to use formal notation and language for functions. Now the input/output relationship is a correspondence between two sets: the domain and the range.[F-IF.1] The domain is the set of input values, and the range is the set of output values. A key advantage of function notation is that the correspondence is built into the notation. For example, $f(5)$ is shorthand for "the output value of f when the input value is 5."

Students sometimes interpret the parentheses in function notation as indicating multiplication. Because they might have seen numerical expressions like $3(4)$, meaning 3 times 4, students can interpret $f(x)$ as f times x. This can lead to false generalizations of the distributive property, such replacing $f(x+3)$ with $f(x)+f(3)$. Work with interpreting function notation in terms of the graph of f can help students avoid this confusion with the symbols (see example in margin).

Although it is common to say "the function $f(x)$," the notation $f(x)$ refers to a single output value when the input value is x. To talk about the function as a whole, write f, or perhaps "the function f, where $f(x) = 3x+4$." The x is merely a placeholder, so $f(t) = 3t+4$ describes exactly the same function.

Later, students can make interpretations like those in the following table:

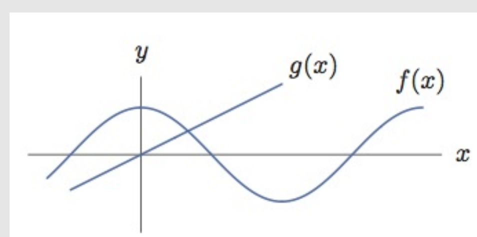
Expression	Interpretation
$f(a+2)$	The output when the input is 2 greater than a
$f(a)+3$	3 more than the output when the input is a
$2f(x)+5$	5 more than twice the output of f when the input is x
$f(b)-f(a)$	The change in output when the input changes from a to b

Notice that a common preoccupation of high school mathematics,

F-IF.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

Interpreting the graph

Use the graph (for example, by marking specific points) to illustrate the statements in (a)–(d). If possible, label the coordinates of any points you draw.



(a) $f(0) = 2$

(b) $f(3) = f(3) = f(9) = 0$

(c) $f(2) = g(2)$

(d) $g(x) > f(x)$ for $x > 2$

For solutions and more discussion of this task, go to Illustrative Mathematics at illustrativemathematics.org/illustrations/636.

The square root function

Since the equation $x^2 = 9$ has two solutions, $x = \pm 3$, students might think incorrectly that $\sqrt{9} = \pm 3$. However, if we want $\sqrt{x}$ to be a function of x, we need to choose one of these square roots. The square root function, $g(x) = \sqrt{x}$, is defined to be the positive square root of x for any positive x.

distinguishing function from relations, is not in the Standards. Time normally spent on exercises involving the vertical line test, or searching lists of ordered pairs to find two with the same x-coordinate and different y-coordinate, can be reallocated elsewhere. Indeed, the vertical line test is problematical, since it makes it difficult to discuss questions such as "is x a function of y" when presented with a graph of y against x (an important question for students thinking about inverse functions). The core question when investigating functions is: "Does each element of the domain corresponds to exactly one element in the range?" The margin shows a discussion of the square root function oriented around this question.

To promote fluency with function notation, students interpret function notation in contexts.[F-IF.2] For example, if h is a function that relates Kristin's height in inches to her age in years, then the statement $h(7) = 49$ means, "When Kristin was 7 years old, she was 49 inches tall." The value of $h(12)$ is the answer to "How tall was Kristin when she was 12 years old." And the solution of $h(x) = 60$ is the answer to "How old was Kristen when she was 60 inches tall?" See also the example in the margin.

Sometimes, especially in real-world contexts, there is no expression (or closed formula) for a function. In those cases, it is common to use a graph or a table of values to (partially) represent the function.

A *sequence* is a function whose domain is a subset of the integers.[F-IF.3] In fact, many patterns explored in grades K–8 can be considered sequences. For example, the sequence 4, 7, 10, 13, 16, ... might be described as a "plus 3 pattern" because terms are computed by adding 3 to the previous term. To show how the sequence can be considered a function, we need an *index* that indicates which term of the sequence we are talking about, and which serves as an input value to the function. Deciding that the 4 corresponds to an index value of 1, we make a table showing the correspondence, as in the margin. The sequence can be describe recursively by the rule $f(1) = 4$, $f(n+1) = f(n) + 3$ for $n \geqslant 2$. Notice that the recursive definition requires both a starting value and a rule for computing subsequent terms. The sequence can also be described with the closed formula $f(n) = 3n + 1$, for integers $n \geqslant 1$. Notice that the domain is included as part of the description. A graph of the sequence consists of discrete dots, because the specification does not indicate what happens "between the dots."

In advanced courses, students may use subscript notation for sequences.

Interpret functions that arise in applications in terms of the context Functions are often described and understood in terms of their *behavior*:[F-IF.4] Over what input values is it increasing, decreasing, or constant? For what input values is the output value positive, negative, or 0? What happens to the output when the input value gets very large positively or negatively? Graphs become very useful

F-IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

Cell Phones

Let $f(t)$ be the number of people, in millions, who own cell phones t years after 1990. Explain the meaning of the following statements.

(a) $f(10) = 100.3$

(b) $f(a) = 20$

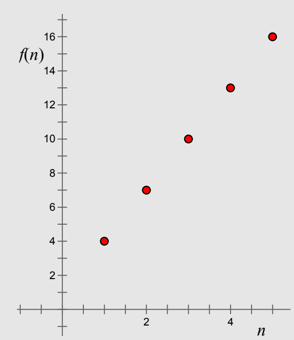
(c) $f(20) = b$

(d) $n = f(t)$

For solutions and more discussion of this task, go to illustrativemathematics.org/illustrations/634.

Sequences as functions

n	$f(n)$
1	4
2	7
3	10
4	13
5	16



F-IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.

F-IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

representations for understanding and comparing functions because these "behaviors" are often easy to see in the graphs of functions (see illustration in margin). Graphs and contexts are opportunities to talk about domain (for an illustration, go to illustrativemathematics.org/illustrations/631).[F-IF.5]

Graphs help us reason about rates of change of function. Students learned in Grade 8 that the *rate of change* of a linear function is equal to the slope of its graph. And because the slope of a line is constant, the phrase "rate of change" is clear for linear functions. For nonlinear functions, however, rates of change are not constant, and so we talk about average rates of change over an interval.[F-IF.6] For example, for the function $g(x) = x^2$, the average rate of change from $x = 2$ to $x = 5$ is

$$\frac{g(5) - g(2)}{5 - 2} = \frac{25 - 4}{5 - 2} = \frac{21}{3} = 7.$$

This is the slope of the line from $(2, 4)$ to $(5, 25)$ on the graph of g. And if g is interpreted as returning the area of a square of side x, then this calculation means that over this interval the area changes, on average, 7 square units for each unit increase in the side length of the square.

Analyze functions using different representations Functions are often studied and understood as families, and students should spend time studying functions within a family, varying parameters to develop an understanding of how the parameters affect the graph of function and its key features.[F-IF.7]

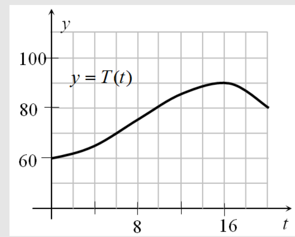
Within a family, the functions often have commonalities in the qualitative shapes of their graphs and in the kinds of features that are important for identifying functions more precisely within a family. This standard indicates which function families should be in students' repertoires, detailing which features are required for several key families. It is an overarching standard that covers the entire range of a student's high school experience; in this part of the progression we merely indicate some guidelines for how it should be treated.

First, linear and exponential functions (and to a lesser extent quadratic functions) receive extensive treatment and comparison in a dedicated group of standards, **Linear and Exponential Models**. Thus, those function families should receive the bulk of the attention related to this standard. Second, all students are expected to develop fluency with linear, quadratic, and exponential functions, including the ability to graph them by hand. Finally, in most of the other function families, students are expected to simple cases without technology, and more complex ones with technology.

F-IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

Warming and Cooling

The figure shows the graph of T, the temperature (in degrees Fahrenheit) over one particular 20-hour period in Santa Elena as a function of time t.



(a) Estimate $T(14)$.

(b) If $t = 0$ corresponds to midnight, interpret what we mean by $T(14)$ in words.

(c) Estimate the highest temperature during this period from the graph.

(d) When was the temperature decreasing?

(e) If Anya wants to go for a two-hour hike and return before the temperature gets over 80 degrees, when should she leave?

For solutions and more discussion of this task, go to Illustrative Mathematics at illustrativemathematics.org/illustrations/639.

F-IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

F-IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

a Graph linear and quadratic functions and show intercepts, maxima, and minima.

b Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

c Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

d (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Consistent with the practice of looking for and making use of structure (MP1), students should also develop the practice of writing expressions for functions in ways that reveal the key features of the function.F-IF.8

Quadratic functions provide a rich playground for developing this ability, since the three principle forms for a quadratic expression (expanded, factored, and completed square) each give insight into different aspects of the function. However, there is a danger that working with these different forms becomes an exercise in picking numbers out of an expression. For example, students often arrive at college talking about "minus b over $2a$ method" for finding the vertex of the graph of a quadratic function. To avoid this problem it is useful to give students translation tasks such as the one in the margin, where they must read both the graphs and the expression and choose for themselves which parts of each correspond.F-IF.9

F-IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

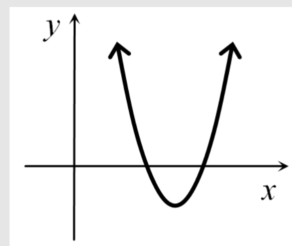
a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

b Use the properties of exponents to interpret expressions for exponential functions.

Which Equation?

Which of the following could be an expression for the function whose graph is shown below? Explain.

(a) $(x+12)^2+4$	(b) $-(x-2)^2-1$
(c) $(x+18)^2-40$	(d) $(x-10)^2-15$
(e) $-4(x+2)(x+3)$	(f) $(x+4)(x-6)$
(g) $(x-12)(-x+18)$	(h) $(20-x)(30-x)$



For solutions and more discussion of this task, go to Illustrative Mathematics at illustrativemathematics.org/illustrations/640.

F-IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

Building Functions

The previous group of standards focuses on interpreting functions given by expressions, graphs, or tables. The Building Functions group focuses on building functions to model relationships, and building new functions from existing functions.

Note: Composition and composition of a function and its inverse are among the plus standards. The following discussion describes in detail what is required for students to grasp these ideas securely. Because of the depth of this development, and because of the subtleties and pitfalls, it is strongly recommended that this content be included only in optional courses.

Build a function that models a relationship between two quantities This cluster of standards is very closely related to the algebra standard on writing equations in two variables.[A-CED.2] Indeed, that algebra standard might well be met by a curriculum in the same unit as this cluster. Although students will eventually study various families of functions, it is useful for them to have experiences of building functions from scratch, without the aid of a host of special recipes, by grappling with a concrete context for clues.[F-BF.1a] For example, in the Lake Algae task in the margin, parts (a)–(c) lead students through reasoning that allows them construct the function in part (d) directly. Students who try a more conventional approach in part (d) of fitting the general function $f(t) = ab^t$ to the situation might well get confused or replicate work already done.

The Algebra Progression discusses the difference between a function and an expression. Not all functions are given by expressions, and in many situations it is natural to use a function defined recursively. Calculating mortgage payment and drug dosages are typical cases where recursively defined functions are useful (see example in the margin).

Modeling contexts also provide a natural place for students to start building functions with simpler functions as components.[F-BF.1bc] Situations of cooling or heating involve functions which approach a limiting value according to a decaying exponential function. Thus, if the ambient room temperature is 70° and a cup of tea made with boiling water at a temperature of 212°, a student can express the function describing the temperature as a function of time using the constant function $f(t) = 70$ to represent the ambient room temperature and the exponentially decaying function $g(t) = 142e^{-kt}$ to represent the decaying difference between the temperature of the tea and the temperature of the room, leading to a function of the form

$$T(t) = 70 + 142e^{-kt}.$$

Students might determine the constant k experimentally.

In contexts where change occurs at discrete intervals (such as payments of interest on a bank balance) or where the input vari-

A-CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

F-BF.1a Write a function that describes a relationship between two quantities.

a Determine an explicit expression, a recursive process, or steps for calculation from a context.

Lake Algae

On June 1, a fast growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area covered by the algae doubles every day. If it continues to grow unabated, the lake will be totally covered and the fish in the lake will suffocate. At the rate it is growing, this will happen on June 30.

(a) When will the lake be covered half-way?

(b) On June 26, a pedestrian who walks by the lake every day warns that the lake will be completely covered soon. Her friend just laughs. Why might her friend be skeptical of the warning?

(c) On June 29, a clean-up crew arrives at the lake and removes almost all of the algae. When they are done, only 1% of the surface is covered with algae. How well does this solve the problem of the algae in the lake?

(d) Write an equation that represents the percentage of the surface area of the lake that is covered in algae as a function of time (in days) that passes since the algae was introduced into the lake if the cleanup crew does not come on June 29.

For solutions and more discussion of this task, go to Illustrative Mathematics at illustrativemathematics.org/illustrations/533.

Drug Dosage

A student strained her knee in an intramural volleyball game, and her doctor has prescribed an anti-inflammatory drug to reduce the swelling. She is to take two 220-milligram tablets every 8 hours for 10 days. Her kidneys filter 60% of this drug from her body every 8 hours. How much of the drug is in her system after 24 hours?

Task from High School Mathematics at Work: Essays and Examples for the Education of All Students (1998), National Academies Press. See http://www.nap.edu/openbook/0309063531/html/80.html for a discussion of the task.

F-BF.1 Write a function that describes a relationship between two quantities.

b Combine standard function types using arithmetic operations.

c (+) Compose functions.

able is a whole number (for example the number of a pattern in a sequence of patterns), the functions chosen will be sequences. In preparation for the deeper study of linear and exponential functions, students can study arithmetic sequences (which are linear functions) and geometric sequences (which are exponential functions).[F-BF.2] This is a good point at which to start making the distinction between additive and multiplicative changes.

Build new functions from existing functions With a basis of experiences in building specific functions from scratch, students start to develop a notion of naturally occurring families of functions that deserve particular attention. It is possible to harden the curriculum too soon around these families, before students have enough experience to get a feel for the effects of different parameters. Students can start getting that feel by playing around with the effect on the graph of simple transformations of the input and output variables.[F-BF.3] Quadratic and absolute value functions are good contexts for getting a sense of the effects of many of these transformations, but eventually students need to understand these ideas abstractly and be able to talk about them for any function f.

Students can get confused about the effect of transformations on the input variable, because the effect on the graph appears to be the opposite to the transformation on the variable. In part (b) of the task in the margin, asking students to talk through the positions of the points in terms of function values can help clear this confusion up.

The concepts of even and odd functions are useful for noticing symmetry. A function f is called an *even function* if $f(-x) = f(x)$ and an *odd function* if $f(-x) = -f(x)$. To understand the names of these concepts, consider that polynomial functions are even exactly when all terms are of even degree and odd exactly when all terms are of odd degree. With some grounding in polynomial functions, students can reason that lots of functions are neither even nor odd.

Students can show from the definitions that the sum of two even functions is even and the sum of two odd functions is odd, and they can interpret these results graphically.

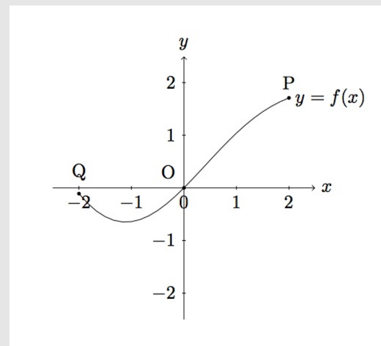
When it comes to inverse functions,[F-BF.4a] the expectations are modest, requiring only that students solve equations of the form $f(x) = c$. The point is to provide an informal sense of determining the input when the output is known. Much of this work can be done with specific values of c. Eventually, some generality is warranted. For example, if $f(x) = 2x^3$, then solving $f(x) = c$ leads to $x = (c/2)^{1/3}$, which is the general formula for finding an input from a specific output, c, for this function, f.

At this point, students need neither the notation nor the formal language of inverse functions, but only the idea of "going backwards" from output to input. This can be interpreted for a table and graph of the function under examination. Correspondences between equa-

F-BF.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

Transforming Functions

The figure shows the graph of a function f whose domain is the interval $-2 \leqslant x \leqslant 2$.



(a) In (i)–(iii), sketch the graph of the given function and compare with the graph of f. Explain what you see.

(i) $g(x) = f(x) + 2$

(ii) $h(x) = -f(x)$

(iii) $p(x) = f(x+2)$

(b) The points labelled Q, O, P on the graph of f have coordinates

$$Q = (-2 - 0.509), \quad O = (0, -0.4), \quad P = (2, 1.309).$$

What are the coordinates of the points corresponding to P, O, Q on the graphs of g, h, and p?

For solutions and more discussion of this task, go to the Illustrative Mathematics website at illustrativemathematics.org/illustrations/742

F-BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

An Interesting Fact

Suppose f is a function with a domain of all real numbers. Define g and h as follows:

$$g(x) = \frac{f(x) + f(-x)}{2} \qquad \text{and} \qquad h(x) = \frac{f(x) - f(-x)}{2}$$

Then $f(x) = g(x) + h(x)$, g is even, and h is odd. (Students may use the definitions to verify these claims.) Thus, any function over the real numbers can be expressed as the sum of an even and an odd function.

F-BF.4a Find inverse functions.

a Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse.

tions giving specific values of the functions, table entries, and point on the graph can be noted (MP1). And although not required in the standard, it is reasonable to include, for comparison, a few examples where the input cannot be uniquely determined from the output. For example, if $g(x) = x^2$, then $g(x) = 5$ has two solutions, $x = \pm\sqrt{5}$.

For advanced mathematics, some students will need a formal sense of inverse functions, which requires careful development. For example, as students begin formal study, they can easily believe that "inverse functions" are a new family of functions, similar to linear functions and exponential functions. To help students develop the instinct that "inverse" is a relationship between two functions, the recurring questions should be "What is the inverse of this function?" and "Does this function have an inverse?" The focus should be on "inverses of functions" rather than a new type of function.

A Joke

Teacher: Are these two functions inverses?
Student: Um, the first one is and the second one isn't.

What does this student misunderstand about inverse functions?

Discussions of the language and notation for inverse functions can help to provide students a sense of what the adjective "inverse" means and mention that a function which has an inverse is known as an "invertible function."

The function $\mathcal{I}(x) = x$ is sometimes called the identity function because it assigns each number to itself. It behaves with respect to composition of functions the way the multiplicative identity, 1, behaves with multiplication of real numbers and the way that the identity matrix behaves with matrix multiplication. If f is any function (over the real numbers), this analogy can be expressed symbolically as $f \circ \mathcal{I} = f = \mathcal{I} \circ f$, and it can be verified as follows:

$$f \circ \mathcal{I}(x) = f(\mathcal{I}(x)) = f(x) \qquad \text{and} \qquad \mathcal{I} \circ f(x) = \mathcal{I}(f(x)) = f(x)$$

Suppose f denotes a function with an inverse whose domain is the real numbers and a is nonzero real number (which thus has a multiplicative inverse), and B is an invertible matrix. The following table compares the concept of inverse function with the concepts of multiplicative inverse and inverse matrix:

Sentence	Interpretation
$f^{-1} \circ f = \mathcal{I} = f \circ f^{-1}$	The composition of f^{-1} with f is the identity function
$a^{-1} \cdot a = 1 = a \cdot a^{-1}$	The product of a^{-1} and a is the multiplicative identity
$B^{-1} \cdot B = \mathcal{I} = B \cdot B^{-1}$	The product of B^{-1} and B is the identity matrix

In other words, where a^{-1} means the inverse of a with respect to multiplication, f^{-1} means the inverse of f with respect to function composition. Thus, when students interpret the notation $f^{-1}(x)$ incorrectly to mean $1/f(x)$, the guidance they need is that the meaning of the "exponent" in f^{-1} is about function composition, not about multiplication.

Notation Inconsistency

In the expression $\sin^2 x$, the exponent is about multiplication, but in $\sin^{-1} x$ the exponent is about function composition. Despite the similar look, these notations use superscripts in different ways. The 2 acts as an exponent but the -1 does not. Both notations, however, allow the expression to be written without the parentheses that would be needed otherwise.

Students do not need to develop the abstract sense of identity and inverse detailed in the above table. Nonetheless, these perspectives can inform the language and conversation in the classroom as students verify by composition (in both directions) that given functions are inverses.[F-BF.4b] Furthermore, students can continue to refine their informal "going backwards" notions, as they consider

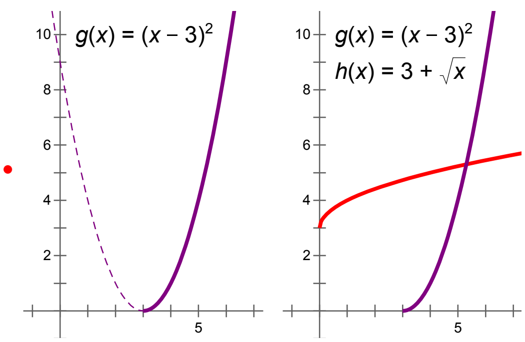
F-BF.4b (+) Verify by composition that one function is the inverse of another.

inverses of functions given by graphs or tables.[F-BF.4c] In this work, students can gain a sense that "going backwards" interchanges the input and output and therefore the stereotypical roles of the letters x and y. And they can reason why the graph of $y = f^{-1}(x)$ will be the reflection across the line $y = x$ of the graph of $y = f(x)$.

F-BF.4c (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

Suppose $g(x) = (x-3)^2$. From the graph, it is easy to see that $g(x) = c$ will have two solutions for any $c > 0$. Thus, to create an invertible function,[F-BF.4d] we must restrict the domain of g so that every range value corresponds to exactly one domain value. One possibility is to restrict the domain of g to $x \geqslant 3$, as illustrated by the solid purple curve in the graph on the left.•

F-BF.4d (+) Produce an invertible function from a non-invertible function by restricting the domain.

• 

When solving $(x-3)^2 = c$, we get $x = 3 \pm \sqrt{c}$, illustrating that positive values of c will yield two solutions x for the unrestricted function. With the restriction, $3 - \sqrt{c}$ is not in the domain. Thus, $x = 3 + \sqrt{c}$, which corresponds to choosing the solid curve and ignoring the dotted portion. The inverse function, then, is $h(c) = 3 + \sqrt{c}$, for $c \geqslant 0$.

We check that h is the inverse of (restricted) g as follows:

$$g(h(x)) = g\left(3 + \sqrt{x}\right) = \left((3 + \sqrt{x}) - 3\right)^2 = (\sqrt{x})^2 = x, \quad x \geqslant 0$$

$$h(g(x)) = h\left((x-3)^2\right) = 3 + \sqrt{(x-3)^2} = 3 + (x-3) = x, \quad x \geqslant 3.$$

The first verification requires that $x \geqslant 0$ so that x is in the domain of h. The second verification requires that $x \geqslant 3$ so that x is in the domain of (restricted) g. And this is precisely what is required for $\sqrt{(x-3)^2}$ to simplify to $(x-3)$.• The rightmost graph shows that the graph of h can be seen as the reflection of the graph of g across the line $y = x$.

• In general, $\sqrt{(x-3)^2} = |x-3|$. If we had chosen the dotted portion of the graph, it would have simplified to $-(x-3)$.

As detailed in the next section, logarithms for most students are merely shorthand for the solutions to exponential equations. Students in advanced classes, however, need to understand logarithms as functions—and as inverses of exponential functions.[F-BF.5] They should be able to explain identities such as $\log_b(b^x) = x$ and $b^{\log_b x} = x$ as well as the laws of logarithms, such as $\log(ab) = \log a + \log b$. In doing so, students can think of the logarithms as unknown exponents in expressions with base 10 (e.g. $\log a$ answers the question "Ten to the what equals a?") and use the properties of exponents,[N-RN.1] building on the understanding of exponents that began in Grade 8.[8.EE.1]

F-BF.5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

N-RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.

8.EE.1 Know and apply the properties of integer exponents to generate equivalent numerical expressions.

Linear and Exponential Models

Construct and compare linear and exponential models and solve problems Distinguishing between situations that can be modeled with linear functions and with exponential functions[F-LE.1a] turns on understanding their rates of growth and looking for indications of these types of growth rates (MP7). One indicator of these growth rates is differences over equal intervals, given, for example, in a table of values drawn from the situation—with the understanding that such a table may only approximate the situation (MP4).

F-LE.1a Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.

To prove that a linear function grows by equal differences over equal intervals,[F-LE.1b] students draw on the understanding developed in Grade 8 that the ratio of the rise and run for any two distinct points on a line is the same (see the Expressions and Equations Progression). An interval can be seen as determining two points on the line whose x-coordinates occur at the boundaries of the intervals. The equal intervals can be seen as the runs for two pairs of points. Because these runs have equal length and ratio between rise and run is the same for any pair of points, the consequence that the corresponding rises are the same. These rises are the growth of the function over each interval.

F-LE.1b Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

Students note the correspondence between rise and run on a graph and differences of inputs and outputs in a symbolic form of the proof (MP1). A symbolic proof has the advantage that the analogous proof showing that exponential functions grow by equal factors over equal intervals begins in an analogous way.

The process of going from linear or exponential functions to tables can go in the opposite direction. Given sufficient information, e.g., a table of values together with information about the type of relationship represented,[F-LE.4], students construct the appropriate function. For example, students might be given the information that the table below shows inputs and outputs of an exponential function, and asked to write an expression for the function.

F-LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Input	Output
0	5
8	33

Interpret expressions for functions in terms of the situation they model Students may build a function to model a situation, using parameters from that situation. In these cases, interpreting expressions for a linear or exponential function in terms of the situation it models[F-LE.5] is often just a matter of remembering how the function was constructed. However, interpreting expressions may be less straightforward for students when they are given an algebraic expression for a function and a description of what the function is intended to model.

F-LE.5 Interpret the parameters in a linear or exponential function in terms of a context.

For example, in doing the task Illegal Fish in the margin, students may need to rely on their understanding of a function as determining an output for a given input to answer the question "Find b if you know the lake contains 33 fish after eight weeks."

See the linear and exponential model section of the Modeling Progression for an example of an interpretation of the intersection of a linear and an exponential function in terms of the situation they model.

Illegal Fish

A fisherman illegally introduces some ïňĄsh into a lake, and they quickly propagate. The growth of the population of this new species (within a period of a few years) is modeled by $P(x) = 5b^x$, where x is the time in weeks following the introduction and b is a positive unknown base.

(a) Exactly how many fish did the fisherman release into the lake?

(b) Find b if you know the lake contains 33 fish after eight weeks. Show step-by-step work.

(c) Instead, now suppose that $P(x) = 5b^x$ and $b = 2$. What is the weekly percent growth rate in this case? What does this mean in every-day language?

For solutions and more discussion of this task, go to Illustrative Mathematics at illustrativemathematics.org/illustrations/579

Trigonometric Functions

Right triangle trigonometry is concerned with ratios of sides of right triangles, allowing functions of angle measures to be defined in terms of these ratios.• This limits the angles considered to those between 0° and 90°. Circular trigonometry extends the domains of the trigonometric functions within the real numbers.

Traditionally, trigonometry includes six functions (sine, cosine, tangent, cotangent, secant, cosecant). Because the second three may be expressed as reciprocals of the first three, this progression discusses only the first three.

• Traditionally, trigonometry concerns "ratios." Note, however, that according to the usage of the Ratio and Proportional Reasoning Progression, that these would be called the "value of the ratio." In high school, students' understanding of ratio may now be sophisticated enough to allow "ratio" to be used for "value of the ratio" in the traditional manner. Likewise, angles are carefully distinguished from their measurements when students are learning about measuring angles in Grades 4 and 5. In high school, students' understanding of angle measure may now allow angles to be referred to by their measures.

Extend the domain of trigonometric functions using the unit circle In circular trigonometry, angles are usually measured in radians rather than degrees. Radian measure is defined so that in the unit circle the measure of an angle is equal to the length of the intercepted arc.F-TF.1 An angle of measure 1 radian turns out to be approximately 57.3°.• A full revolution, which corresponds to an angle of 360°, has measure equal to the circumference of the unit circle, or 2π. A quarter turn, or 90°, measures $\pi/2$ radians.

In geometry, students learn, by similarity, that the radian measure of an angle can be defined as the quotient of arc length to radius.G-C.5 As a quotient of two lengths, therefore, radian measure is "dimensionless." That is why the "unit" is often omitted when measuring angles in radians.

In calculus, the benefits of radian measure become plentiful, leading, for example, to simple formulas for derivatives and integrals of trigonometric functions. Before calculus, there are two key benefits of using radians rather than degrees:

- arclength is simply $r\theta$, and
- $\sin\theta \approx \theta$ for small θ.

F-TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

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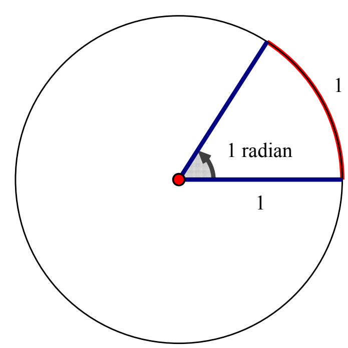

G-C.5 Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

In right triangle trigonometry, angles must be between 0° and 90°. Circular trigonometry allows angles that describe any amount of rotation, including rotation greater than 360°.F-TF.2 Consistent with conventions for measuring angles, counterclockwise rotation is associated with angles of positive measure and clockwise rotation is associated with angles of negative measure. The topic is called circular trigonometry because all rotations take place about the origin on the circle of radius 1 centered at the origin (called the unit circle), in the the coordinate plane.

F-TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

With the help of a picture,• students mark the intended angle, θ, in radians, measured counterclockwise from the positive ray of the x-axis; identify the coordinates x and y; draw a reference triangle; and then use right triangle trigonometry. In particular, $\sin\theta = y/1 = y$, $\cos\theta = x/1 = x$, and $\tan\theta = y/x$. (Note the simplicity afforded by using a circle of radius 1.) This way, students can compute values of any of the trigonometric functions, except that in circular

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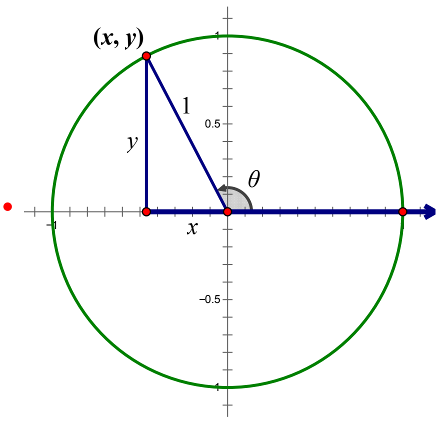
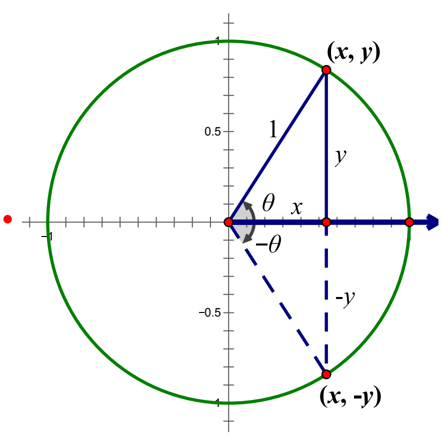


trigonometry being careful to note the signs of x and y. In the figure as drawn in the second quadrant, for example, x is negative and y is positive, which implies that $\sin\theta$ is positive and $\cos\theta$ and $\tan\theta$ are both negative.

The next step is sometimes called "unwrapping the unit circle." On a fresh set of axes, the angle θ is plotted along the horizontal axis and one of the trigonometric functions is plotted along the vertical axis. Dynamic presentations with shadows can help considerably, and the point should be that students notice the periodicity of the functions, caused by the repeated rotation about the origin, regularly reflecting on the grounding in right triangle trigonometry.

With the help of the special right triangles, 30°-60°-90° and 45°-45°-90°, for which the quotients of sides are easily computed, the values of the trigonometric functions are easily computed for the angles $\pi/3$, $\pi/4$, and $\pi/6$ as well as their multiples.[F-TF.3] For advanced mathematics, students need to develop fluency with the trigonometric functions of these special angles to support fluency with the "unwrapping of the unit circle" to create and graph the trigonometric functions.

F-TF.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosines, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.



Using symmetry, students can see that, compared to the reference triangle with angle θ, an angle of $-\theta$ will produce a congruent reference triangle that is its reflection across the x-axis. They can then reason that $\sin(-\theta) = -y = -\sin(\theta)$, so sine is an odd function. Similarly, $\cos(-\theta) = x = \cos(\theta)$, so cosine is an even function.[F-TF.4] Some additional work is required to verify that these relationships hold for values of θ outside the first quadrant.

The same sorts of pictures can be used to argue that the trigonometric functions are periodic. For example, for any integer n, $\sin(\theta + 2n\pi) = \sin(\theta)$ because angles that differ by a multiple of 2π have the same terminal side and thus the same coordinates x and y.

F-TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Model periodic phenomena with trigonometric functions Now that students are armed with trigonometric functions, they can model some periodic phenomena that occur in the real world. For students who do not continue into advanced mathematics, this is the culmination of their study of trigonometric functions.

The tangent and function is not often useful for modeling periodic phenomena because $\tan x$ is undefined for $x = \frac{\pi}{2} + k\pi$, where k is an integer. Because the sine and cosine functions have the same shape, either suffices to model simple periodic phenomena.[F-TF.5] Functions are called *sinusoidal* if they have the same shape as the sine graph. Many real-world phenomena can be approximated by sinusoids, including sound waves, oscillation on a spring, the motion of a pendulum, tides, and phases of the moon. Some students will learn in college that sinusoids are used as building blocks to approximate any periodic waveform.

F-TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

The general sinusoid is given by $f(t) = A + B\sin(Ct + D)$. Students can reason that because $\sin(\theta)$ oscillates between -1 and 1,

$A+B\sin(Ct+D)$ will oscillate between $A-B$ and $A+B$. Thus, $y = A$ is the midline, and B is the amplitude of the sinusoid. Students can obtain the frequency from this equation by noting the period of $sint$ of 2π and knowing the frequency is the reciprocal of the period, so the period of $sinCt$ is $2\pi/C$. When modeling, students need to have the sense that C affects the frequency and that C and D together produce a phase shift, but getting these correct might involve technological support, except in simple cases.

For example, students might be asked to model the length of the day in Columbus, Ohio. Day length as a function of date is approximately sinusoidal, varying from about 9 hours, 20 minutes on December 21 to about 15 hours on June 21. The average of the maximum and minimum gives the value for the midline, and the amplitude is half the difference. So $A \approx 12.17$, and $B \approx 2.83$. With some support, students can determine that for the period to be 365 days (per cycle), $C = 2\pi/365$, and if day 0 corresponds to March 21, no phase shift is needed, so $D = 0$. Thus,

$$f(t) = 12.17 + 2.83\sin\left(\frac{2\pi t}{365}\right)$$

From the graph, students can see that the period is indeed 365 days, as desired, as it takes one year to complete the cycle. They can also see that two days are approximately 14 hours long, which is to say that $f(t) = 14$ has two solutions over a domain of one year, and they might use graphing or spreadsheet technology to determine that May 1 and August 10 are the closest such days. Students can also see that $f(t) = 9$ has no solutions, which makes sense because 9 hours, 20 minutes is the minimum length of day.

Students who take advanced mathematics will need additional fluency with transformations of trigonometric functions, including changes in frequency and phase shifts.

Based on plenty of experience solving equations of the form $f(t) = c$ graphically, students of advanced mathematics should be able to see that such equations will have an infinite number of solutions when f is a trigonometric function. Furthermore, they should have had experience of restricting the domain of a function so that it has an inverse. For trigonometric functions, a common approach to restricting the domain is to choose an interval on which the function is always increasing or always decreasing.[F-TF.6] The obvious choice for $\sin(x)$ is the interval $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, shown as the solid part of the graph. This yields a function $\theta = \sin^{-1}(x)$ with domain $-1 \leqslant x \leqslant 1$ and range $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$.

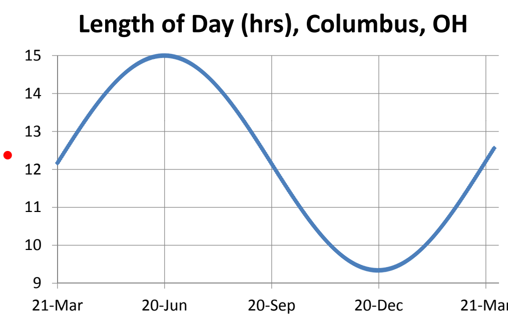
Inverses of trigonometric functions can be used in solving equations in modeling contexts.[F-TF.7] For example, in the length of day context, students can use inverse trig functions to determine days with particular lengths. This amounts to solving $f(t) = d$ for t, which yields

$$t = \frac{365}{2\pi}\sin^{-1}\left(\frac{d - 12.17}{2.83}\right)$$

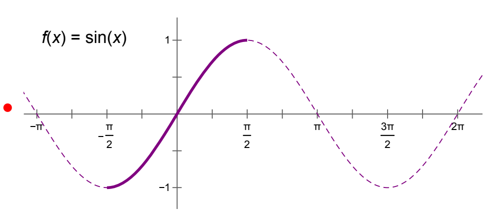
Frequency vs. Period

For a sinusoid, the frequency is often measured in cycles per second, thus the period is often measured in seconds per cycle. For reasoning about a context, it is common to choose whichever is greater numerically.

or for the frequency to be $\frac{1}{365}$ cycles/day



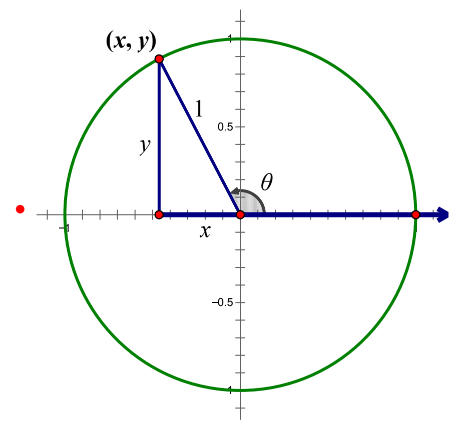
F-TF.6 (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.



F-TF.7 (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.

Using $d = 14$ and a calculator (in radian mode), they can compute that $t \approx 40.85$, which is closest to May 1. Finding the other solution is a bit of a challenge, but the graph indicates that it should occur just as many days before midyear (day 182.5) as May 1 occurs after day 0. So the other solution is $t \approx 182.5 - 40.85 = 141.65$, which is closest to August 10.

Prove and apply trigonometric identities For the cases illustrated by the diagram (in which the terminal side of angle θ does not lie on an axis) and the definitions of $\sin\theta$ and $\cos\theta$, students can reason that, in any quadrant, the lengths of the legs of the reference triangle are $|x|$ and $|y|$. It then follows from the Pythagorean Theorem that $|x|^2 + |y|^2 = 1$. Because $|a|^2 = a^2$ for any real number a, this equation can be written $x^2 + y^2 = 1$. Because $x = \cos\theta$ and $y = \sin\theta$, the equation can be written as $\sin^2(\theta) + \cos^2(\theta) = 1$. When the terminal side of angle θ does lie on an axis, then one of x or y is 0 and the other is 1 or -1 and the equation still holds. This argument proves what is known as the Pythagorean identity[F-TF.8] because it is essentially a restatement of the Pythagorean Theorem for a right triangle of hypotenuse 1.



F-TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

With this identity and the value of one of the trigonometric functions for a given angle, students can find the values of the other functions for that angle, as long as they know the quadrant in which the angle lies. For example, if $\sin(\theta) = 0.6$ and θ lies in the second quadrant, then $\cos^2(\theta) = 1 - 0.6^2 = 0.64$, so $\cos(\theta) = \pm\sqrt{0.64} = \pm 0.8$. Because cosine is negative in the second quadrant, it follows that $\cos(\theta) = 0.8$, and therefore $\tan(\theta) = \sin(\theta)/\cos(\theta) = 0.6/(-0.8) = -0.75$.

Students in advanced mathematics courses can prove and use other trigonometric identities, including the addition and subtraction formulas.[F-TF.9] If students have already represented complex numbers on the complex plane[N-CN.4] and developed the geometric interpretation of their multiplication,[N-CN.5] then the the product $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$ can be used in deriving the addition formulas for cosine and sine. Subtraction and double angle formulas can follow from these.

F-TF.9 (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

N-CN.4 (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

N-CN.5 (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation.